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SHAPE NUMBERS: A NOTATION TO DESCRIBE PURE FORM AND TO MEASURE RESEMBLANCE AND DIFFERENCE IN SHAPE*

Ernesto Bribiesca**
Adolfo Guzmán***

*Reporte Técnico PR-78-20, Laboratorio PR.

**Investigador de la Dirección General de Estudios del Territorio Nacional (antes CETENAL).

***Investigador del IIMAS.

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Abstract

A description is derived for two-dimensional non-intersecting closed curves that are the boundary of simply connected regions. This description is independent of their size, orientation and position, but it depends on their shape; it is therefore called the shape number of the curve.

Each curve carries with it its own shape number. The order of the shape number indicates the precision with which that number describes the shape of the curve. For a curve, the order of its shape number is the length of the perimeter of a 'discrete shape' (a closed curve formed by vertical and horizontal segments, all of equal length) closely corresponding to the curve.

A procedure is given that deduces, without table look-up, string matching or correlations, the shape number of any order for an arbitrary curve.

In this way, the infinite universe of curves can be decomposed, for any order o, into a finite number of equivalence classes, each one of them having the same shape number of order o. The discrete shapes stand as the canonical representative for each class. The paper contains all the families of discrete shapes for orders 4, 6, 8, 10, 12, 14 and 16.

To find out how close in shapes two curves are, the degree of similarity between them is introduced; dissimilar regions will have it low, while analogous shapes will have a high degree of similarity. Informally speaking, the degree of similarity between the shapes of two curves tells how deep you have to descend into a list of shapes, before being able to differentiate between the shape of those two curves. Again, a procedure is given to compute it, without need for such list or grammatical parsing or least square curve or area fitting.

The degree of similarity maps the universe of curves into a tree or hierarchy of shapes. The distance between the shapes of any two curves, defined as the inverse of their degree of similarity, is found to be an ultradistance over this tree.

The shape number is a description that changes with skewing, anisotropic dilation and mirror images, as the intuitive psychological concept of 'shape' demands. Nevertheless, at the end of the paper a related Theory "B" of shapes is introduced that allows anisotropic changes of scale, thus permitting for instance a rectangle and a square to have the same Bshape.

These definitions and procedures may facilitate a quantitative study of shape.

Key words: curve description; chain encoding; shape code; silouettes; shape numbers; distance between shapes; form similarity; discrete shapes; shape comparison; measure of shape difference; binary picture; image processing.

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CONTENTS

INTRODUCTION

The usefulness of picture analysis. Picture analysis and photointerpretation are very useful tools that provide information of widespread value: creation of maps [N.A. Bryant] [these brackets indicate references to the bibliography]; estimation of crops [Landgreebe; MacDonald; Guzmán et al 76]; tectonics [Salas]; electrocardiography [González], to mention just a few. It is not surprising, consequently, that computers are increasingly used to aid in this process: water detection [Wright ; Seco]; plaques in blood vessels [Selzer]; digital terrain models [Peucker; Dora Gómez] are a few examples where a computer extracts useful information from a picture.

Color, texture, stereoscopy and shape. Crop detection and land use maps are examples where a computer program [MacDonald ; Guzmán PR-75-2A] uses color as a primary ingredient for identification. The texture of a given zone in a picture (that is, the spatial relation and structure of small regions of peculiar shapes) carries also valuable information; for instance, terrain drainage [Felipe Guerra]; citric trees diseases [Mancillas].

Three dimensional information can be obtained from two or more views of the same scene: assemblies of bodies [Roberts]; tomography [Hernan]; contour lines [D. Gómez].

In this paper, we concentrate on the shape of objects as the main ingredient to extract information that will allow us to interpret the scene. Of course, it is realized that a system that uses several or all of these ingredients will obtain better information [Reddy].

Coloring book drawings. Since we decided to study shape, what images are color-less, texture-less and monoscopic? It has been proposed [Guzmán 71] to analyze line drawings such as those found in coloring books for children (See Figure 'SKATING'), because they are only binary (black and white) scenes.

Although these scenes are hand-made (as opposed to camera-made), they retain the shape information of the objects. Since small children can understand them, we hope that they should not be too difficult for the machine, either!

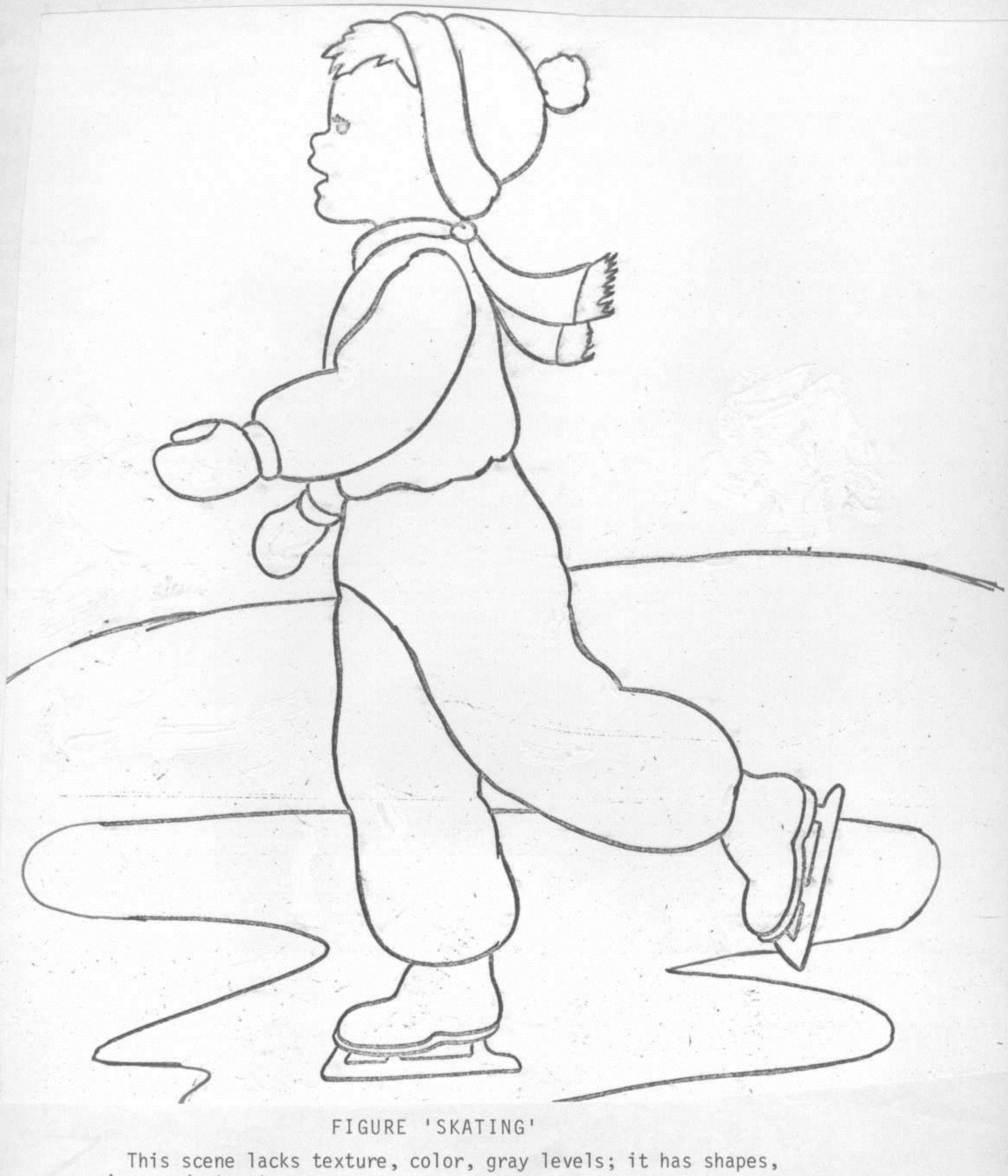


FIGURE 'SKATING'

This scene lacks texture, color, gray levels; it has shapes, sizes and structure.

The shape of each region of this scene can be described by a shape number.

Caricatures [Adler] are not used because they contain shape distortions, although we realize that they shine light over the permissible transformations of shape that preserve information used for identification by human beings.

We are not recommending that somebody should work on a preprocessor that will transform gray level pictures into coloring book drawings.

The role of shape in Scene Analysis

A good explanation and understanding can be done of figure 'SKATING', which lacks color, texture, and gray levels and only has shapes with size and structure. [Guzmán 71] has proposed to represent these components into a graph where the nodes contain shape and size information about each region, and the arcs represent structural relations ("near", "surrounded by", etc.) among the nodes.

Consequently, it is important to be able to describe in a convenient manner the shape or form of a region (or part, or object), and to compare two shapes in order to ascertain their likeness or dissimilarity.

The quantification of those concepts through numerical procedures yielding repeatable and reliable measures is part of the quantitative study of shape.

DISCRETE NOTATIONS TO ENCODE LINES AND REGIONS

Region (def). A simply connected portion of a plane limited by a curve boundary. That is, no holes, no self-intersecting boundary. Closed boundary. The region is uniquely defined by the curve it has as boundary.

A region has size (length of the perimeter, area of its surface, ...), position (in the plane), orientation (with respect to some coordinate system), as well as "aspect", form or shape.

Our notation will describe regions. It will describe shapes also; that is, in fact, its main use.

Shape (def). A region where its size, position and orientation are disregarded.

Two regions have the same shape if through a similarity transformation (translation, rotation, uniform stretching of both axis) they can coincide exactly.

Note that mirror images do not have in general the same shape. Skewing (to change the angle between the X and Y axis) is also not allowed. Neither is permissible an anisotropic stretching of the axis, i. e. $a \neq b$ in $X' = a\,x$, $Y' = b\,y$. We later introduce a theory "B" of shapes where it is permitted to have $a \neq b$; their shapes are called Bshapes.

Scene (def). A collection of regions.

We generally use the same scenes that children use for coloring, like figure 'SKATING.'

Since regions posses position, size, etc., a scene is "rigid" and we can measure in it orientations, relative positions, relations between regions, etc.

Note that open lines are not allowed in a scene. This may be quite a restriction, specially in view of the fact that our notation is able to describe them.

Major axis of a region. The straight line segment connecting the two perimeter points furthest away from each other. Figure 'BASIC RECTANGLE.'

Occassionally, there will be more than one major axis in the region.

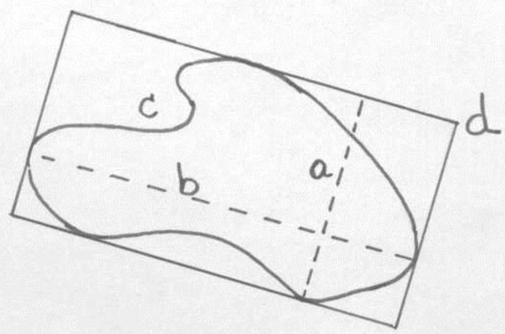


FIGURE 'BASIC RECTANGLE'

a. Minor axis of (c).
b. Major axis of (c).
c. Region.
d. Basic rectangle of (c).

In that case, select that which gives the shortest minor axis.

Minor axis of a region. A segment perpendicular to the major axis, extending in both sides of the major axis, and of length such that the box formed by these two axis just encloses the region. Figure 'BASIC RECTANGLE.'

Basic rectangle of a region. It is the rectangle having its sides parallel to and of sizes equal to the major and minor axis, such that it just encloses the region.

Other axis and other manners to find boxes enclosing a region are given in [Freeman and Shapira] and in [Guzmán 71], pp 338-342.

Excentricity of a rectangle. It is the ratio of the long to the short side. e is greater than or equal to one.

Excentricity of a region. It is the excentricity of its basic rectangle.

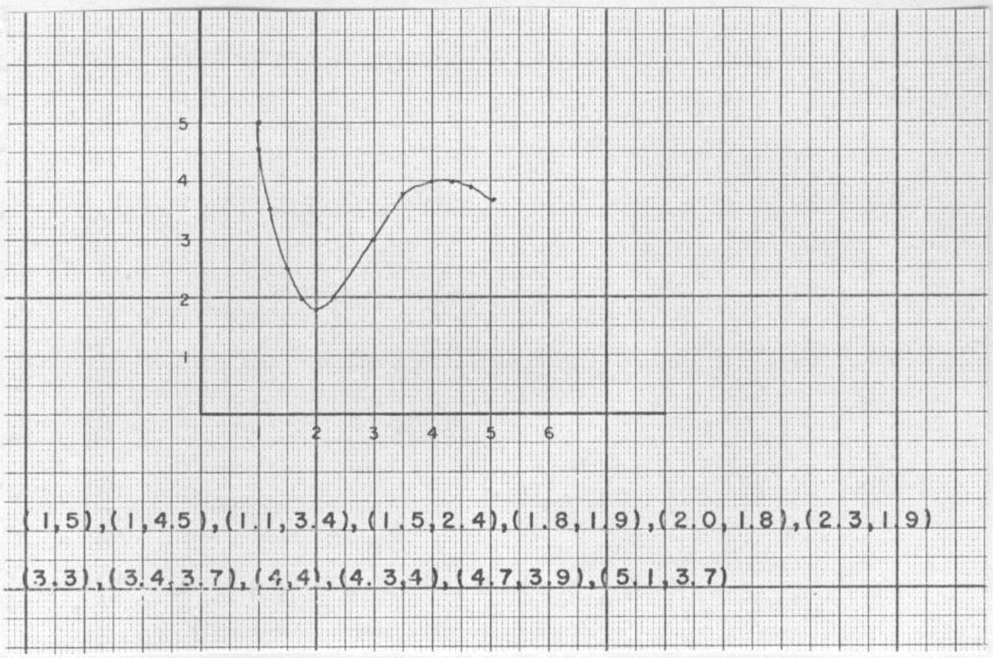
It is the ratio of its major to minor axis. This definition coincides with that for an ellipse.

Digital representation of curves

We now study several ways to represent figures. To keep memory finite, we sample or discretize the curve in some manner. This allows a digital (finite size, finite memory) representation, but usually from such representation the original curve can be recovered only approximately, due to the quantization error (pixel size).

There is no "best representation" in general. Each one is more suitable than others for certain purposes. We are looking for representations useful to compare shapes: are these two figures of the same shape? Is A closer in shape to B, or to C?

X,Y representation. This is one of the simplest representations. We select arbitrary points (many and close together, if we want good accuracy) on the curve, and we write down the X,Y coordinates of each of them, as we travel along it.



Coordinates of the points: (1.0, 5.0), (1.0, 4.5), (1.1, 3.4), (1.5, 2.4), (1.8, 1.9), (2.0, 1.8), (2.3, 1.9), (3.0, 3.0), (3.4, 3.7), (4.0, 4.0), (4.0, 3.4), (4.7, 3.9), (5.1, 3.7).

X,Y representation of the curve:
1050104511341524181920182319303034374040403447395137

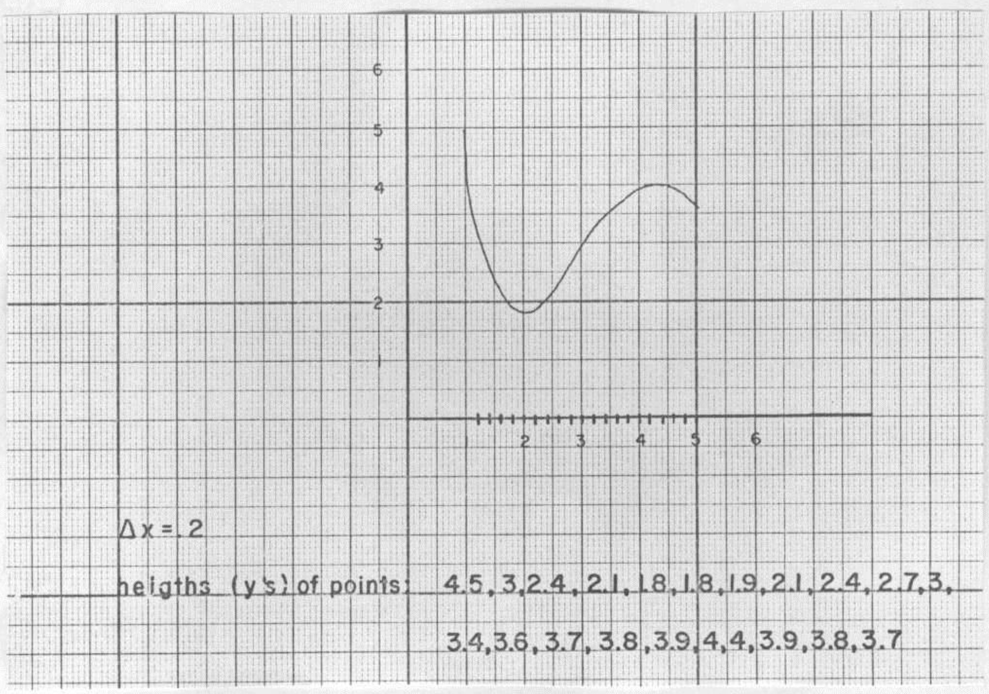
Advantages of this representation: easy to obtain, easy to reconstruct. Disadvantages: difficult to compare shape. Uneven signal to noise ratio. Curves not uniquely coded: difficult to compare whether representation a and representation b represent indeed the same curve.

There are many variants to this representation. We show a few.

Select an uniform Δx. In this case, we begin the representation with the value of Δx, which can also be omited, if it is fixed and known for all our figures.

Advantage: No need to write the x coordinate values.

Most digital recording instruments sample at fixed time intervals, and therefore use this method, where x is the time.



$\Delta x = 0.2$

Heigths (Y's) of points: 4.5, 3.0, 2.4,2.1, 1.8, 1.8, 1.9, 2.1, 2.4, 2.7, 3.0, 3.4, 3.6, 3.7, 3.8, 3.9, 4.4, 3.9, 3.8, 3.7.

Representation of the curve:

45302421181819212427303436373839444393837.

Always place the origin (0,0) at the first point. This uses less bits if the values of the coordinates are large (if the origin was far away).

Use Δy instead of y. Above, instead of writing down the values of (x,y), or of y only if Δx is fixed, write down the first value of y, and after it, only increments and decrements of y. This method saves bits if the values of y are large.

The method, in conjunction with an uniform Δx, is called delta modulation. See below.

Example: heigth (y) of first point: 4.5

Increments of succesive points with respect to the previous point (that closest to its left):

-1.5, -0.6, -0.3, -0.3, 0.0, 0.1, 0.2, 0.3, 0.3, 0.3, 0.4, 0.2, 0.1, 0.1, 0.1, 0.5, -0.5, -0.1, -0.1.

Representation of the curve:

-15-06-03-03+00+01+02+03+03+03+04+02+01+01+01+05-05-01-01

Since there will be positive and negative numbers, the signs + and - are necessary. They add an extra bit to every coordinate.

Use sgn(Δy) instead of Δy. For slowly varying signals; transmit (encode) +1 if the signal is going up, -1 otherwise. Optional: transmit 0 if it has remained approximately constant. This is delta modulation [Steel]. Disadvantage: the representation slowly "catches up" with fast varying parts.

Delta modulation using two or more bits. Transmit as follows:

00 if the signal is going up

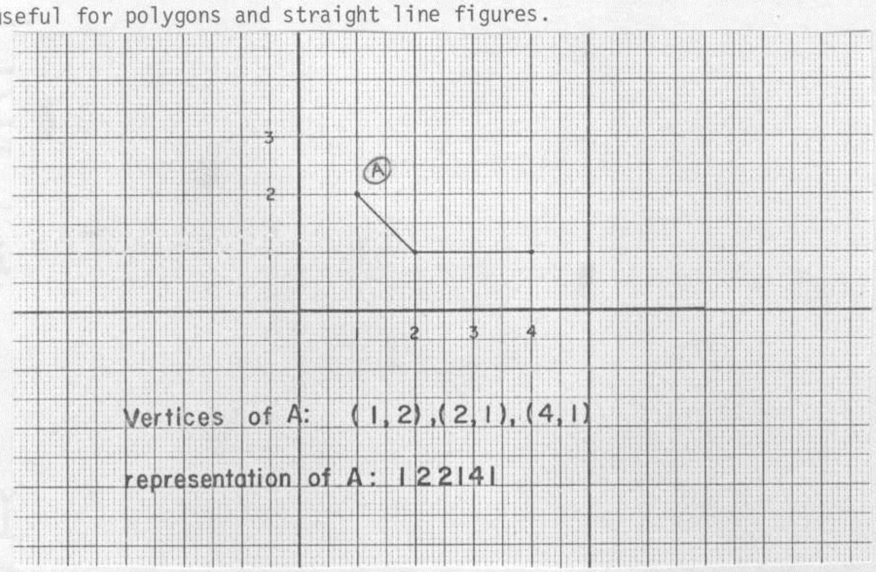
01 if the signal has remained at the same level

10 if the signal is going down slowly.

11 if the signal is going down fast.

You could use more bits, for more accuracy; also, the exact meaning of "signal is going down slowly" has to be given.

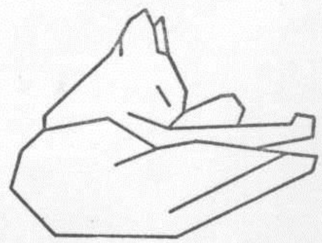
Do not use uniformly distributed points. Place them at the vertices. This is useful for polygons and straight line figures.



Vertices of A: (1,2), (2,1), (4,1).
Representation of A: 1 2 2 1 4 1

Concentrate the points at places where curvature is high. This method has the advantage of keeping the error more constant [O'Callaghan]. Also, there is psychological evidence [Attneave] that people use it; see Figure 'SLEEPING CAT.'

The method can become quite ellaborate and useful; as when [Gómez] describes three-dimensional surfaces through especialized points, giving as a result a surface model (digital model of the surface or terrain) which recursively guides the procedure that finds points on the surface for inclusion of them into the model. The model guides the construction of itself. The advantages are few points on the model and a signal to noise ratio nearly constant over the 3-d surface.



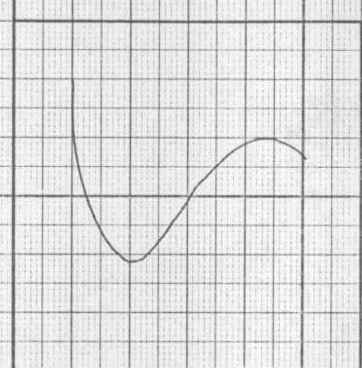
A sleeping cat, drawn by connecting the points of maximum curvature in its outline.

(From Attneave, 1954.)

FIGURE 'SLEEPING CAT'

Do not join the points with straight lines. Use low order polynomial expressions to interpolate several points. Only when the error exceeds some limit, introduce another point. This method requires fewer points, but construction and reconstruction of the curve in slow

Binary matrix representation. Use a retina (an array of cells) and on each pixel where the line falls, place a 1. Elsewhere place a 0. Perceptrons /Rosenblatt; Minsky and Papert/ use this representation. Example:

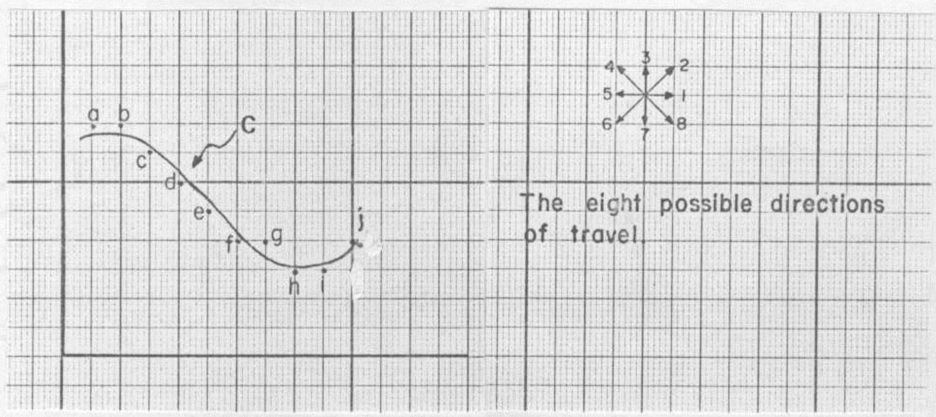


representation:

0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	1	1	1	1	0
0	0	1	0	0	0	1	0	0	0	0	0
0	0	1	0	0	1	0	0	0	0	0	0
0	0	1	1	1	1	0	0	0	0	0	0
0	0	0	1	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0

Most optical scanners use this method, but instead of a binary picture they may give a matrix with gray level values at each pixel (each cell).

Freeman chains. On top of the curve place a grid. At each crossing of the curve with a line of the grid, choose the closest node of the grid. This defines a set of grid points near the curve in question. Now begin travelling these points from the first (that corresponding to the beginning of the curve) to the last, taking note of the directions (one of eight possible) of movement [Freeman].



C is the curve, and a, b, ..., j the points of the grid closest to it.

We start at a. To go from a to b, you move in the direction →1. To go from b to c, you move in the direction ↘8, and so on.

The Freeman chain is 1 8 8 8 8 1 8 1 2.

Other figures and their Freeman chains are given in the next page.

Going from Freeman chains to (x, y) representation. Begin at an arbitrary origin. Add to it the amounts shown in the table 'Going from Freeman chains to (x,y) representation.'

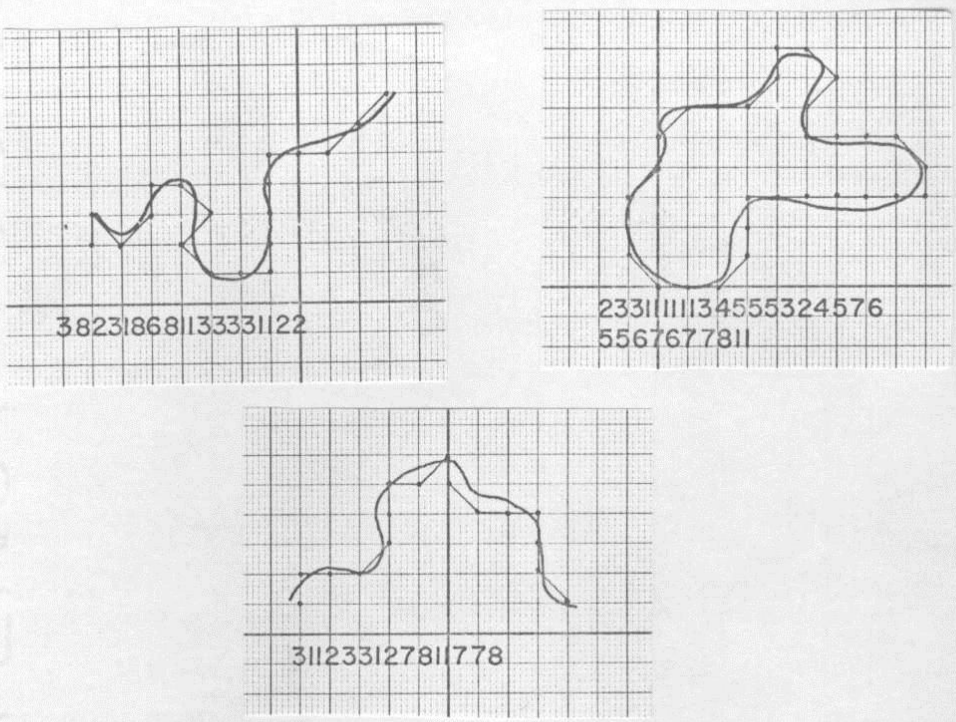


FIGURE 'FREEMAN CHAIN OF CURVES'

<u>Going from (x,y) representation to Freeman chains</u>. Begin with the points in the grid (at the nodes of the grid) closest to the curve in question. Follow the table 'Going from (x,y) representation to Freeman chains.' An example is also given in that table.

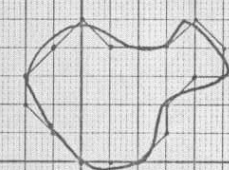
<u>How to change accuracy in Freeman chains</u>. Just change the size of the grid, and recompute the directions accordingly. Observe the examples given in the table 'How to change accuracy in Freeman chains.'

[Bribiesca and Avilés] have developed methods to change representation, accuracy and other things for lines encoded by chains.

Going From Freeman chains to (x,y) representation.

Table

chain's direction	Δx	Δy
1	1	0
2	1	1
3	0	1
4	-1	1
5	-1	0
6	-1	-1
7	0	-1
8	1	-1



1 1 2 3 2 1 3 4 6 5 5 4 6 6 7 8 8

Using the table.

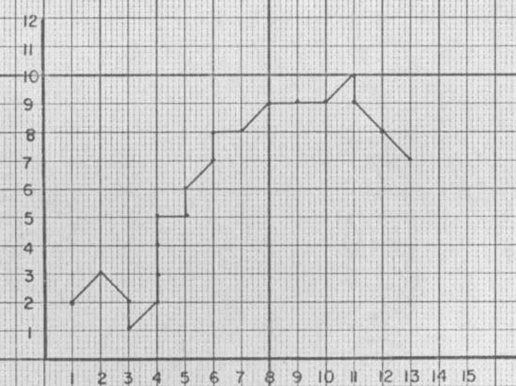
(0,0) (1,0) (2,0) (3,1) (3,2) (4,3) (5,3) (5,4) (4,5) (3,4)

(2,4) (1,4) (0,5) (-1,4) (-2,3) (-2,2) (-1,1) (0,0)

Going from (x,y) representation to freeman chain

Table

Δx	Δy	chain's direction
1	0	1
1	1	2
0	1	3
-1	1	4
-1	0	5
-1	-1	6
0	-1	7
1	-1	8



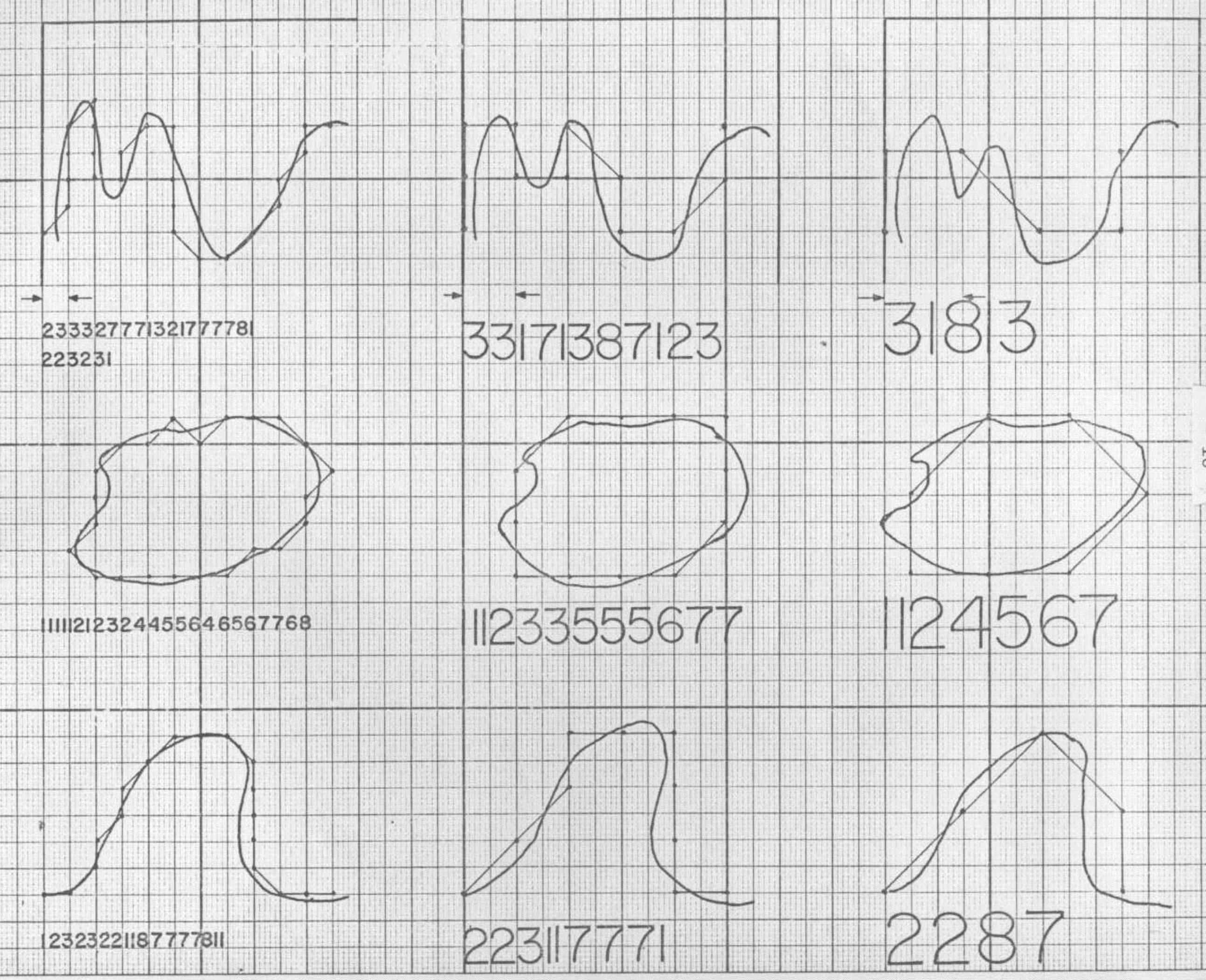
(1,2) (2,3) (3,2) (3,1) (4,2) (4,3) (4,4) (4,5) (5,5) (5,6) (6,7) (6,8)

(7,8) (8,9) (9,9) (10,9) (11,10) (11,9) (12,8) (13,7)

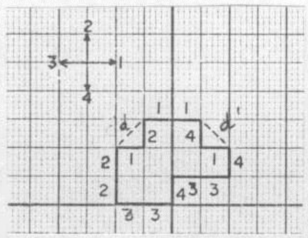
Using the Table.

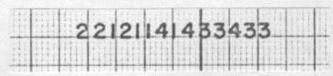
2872333132312112788

How to change accuracy in freeman chins.



Freeman chains in four directions. Use four instead of eight directions:

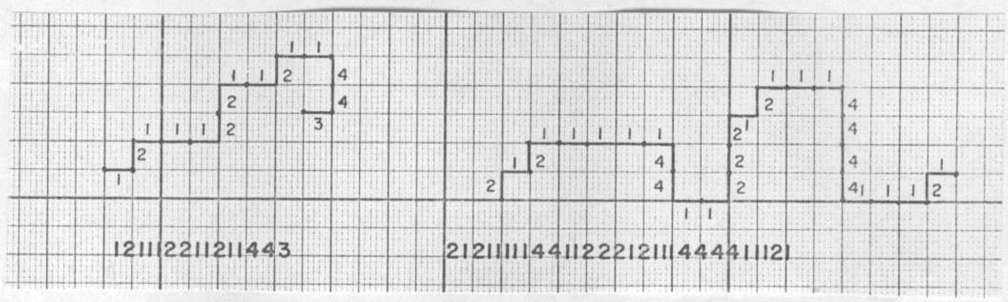


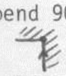
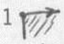


Note that the directions d and d' are represented by 1 2 or 4 1. Select one or the other after seeing whether the corresponding square (of the grid) contains more than or less than 50 % of the area of the region that the curve encloses (for closed curves).

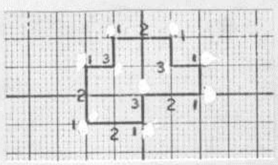
For closed curves, an alternative definition for Freeman chains in four direction is "the curve obtained by walking clockwise on the "wires" of the grid around and outside the squares that contain more than 50 per cent of the region." This definition gives slightly different results from the former, because one is based in squares filled more than half; the other sees the nodes of the grid that come closest to the curve.

In this paper the "50 %" definition will be used for closed curves. This could be slower (because it measures areas) than the other definition, which frequently finds the node of the grid closest to a given segment of the curve by truncation of coordinate values. Some examples are:



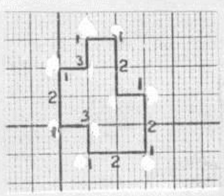
Derivative of a Freeman chain. The idea is to refer the direction of the next link (in the chain) to the direction of the current link. We use four directions. Thus, to bend 90 degrees to the left (keeping the region to the right) is coded as 3:  3 To go straight is a 2: 
To turn 90 degrees to the right is coded as 1: 1 

Notice that the bents (turns, changes of direction) or their absence are coded at each grill point. This makes the coding to lose (that is, not to encode) the orientation of the figure. When we write the chain, we are travelling clockwise on the boundary, having the region to our right. Let us see the figure of the previous page, encoded now in Derivative of Freeman chain:



12132113121312

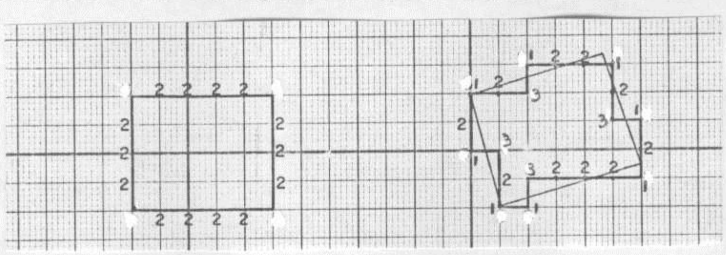
and if the figure is rotated 90 degrees, the coding is still the same:



12132113121312

That is, the coding does not preserve orientation.

This code is not quite invariant with respect to rotations, because of the distortions arising when we rotate digital pictures. Observe:

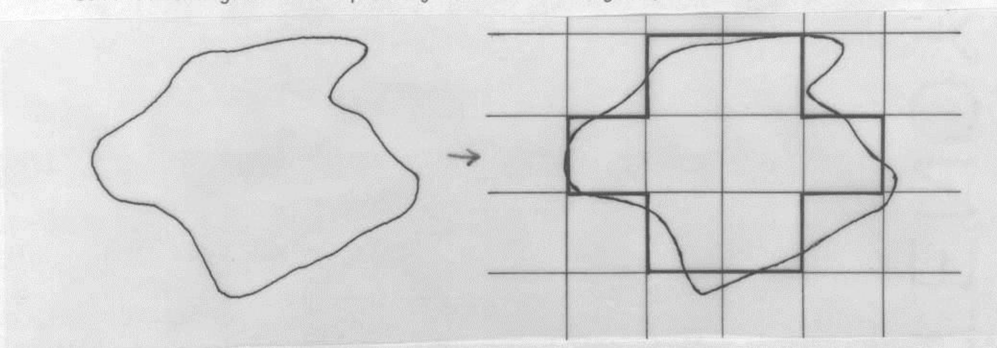


1222212221222212222 1132221213212213212132

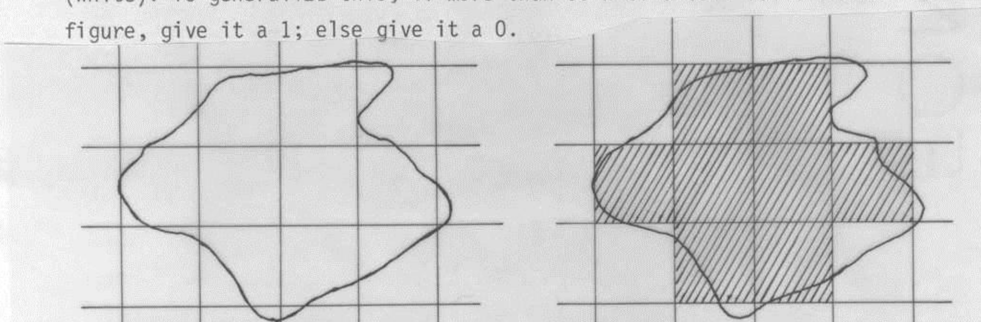
THE SHAPE NUMBERS

We finally arrive to our proposed description of the shape of <u>shapes</u> and <u>regions</u>. The procedure to find the shape number of a region is as follows:

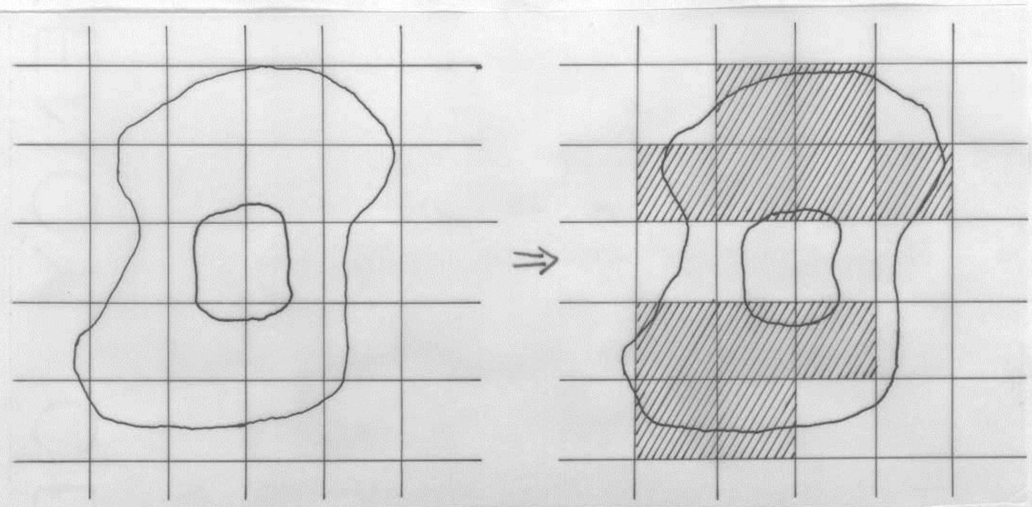
1. A grid of arbitrary cell size is overlaid on top of the region. If a cell of the grid is completely inside the figure, it becomes marked with



a 1 (black); if it falls completely outside the figure, it receives a 0 (white). To generalize this, if more than 50 % of a cell falls inside the figure, give it a 1; else give it a 0.



If the figure has holes, it is probable that none of the squares of a row or column reach a 1, in which case we obtain a blank row (or column) that divides the figure in two:



In these cases, we lower the acceptance level (for instance, 35% instead of 50% of a square inside the region will produce a 1). (A better way is to declare that the shape number does not exist; see this later, specially with respect to Theory "B" of shapes). But this test for a blank row or column, which the program performs now, needs to be generalized to the test for disjoint regions:

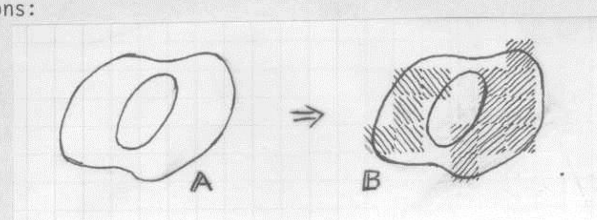
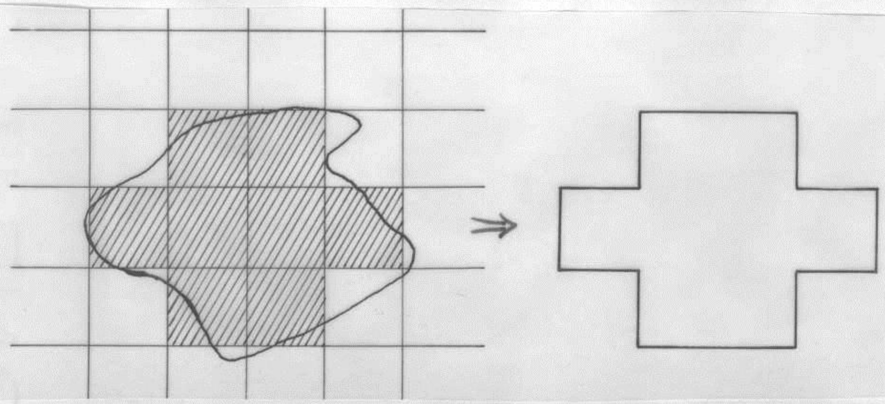
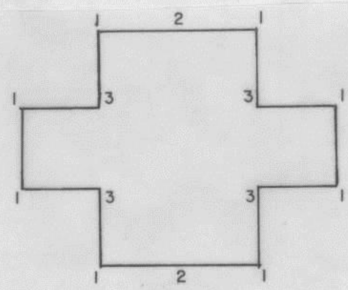


Figure B is disjoint but none of its columns or rows is completely blank (0's alone). This problem is solved later, cf. § 'Razonable shape' and Fig. 'Holes and degenerate shapes'.

Now, the boundary of such new black region is the chain sought after:



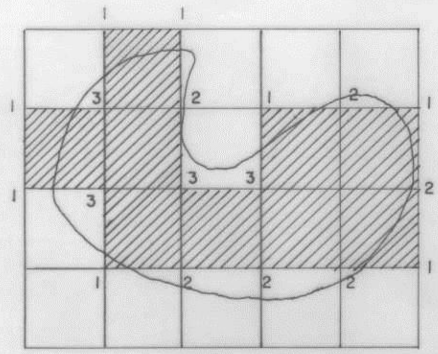
2. We denote this chain by its derivative notation (q.v.). The result is:



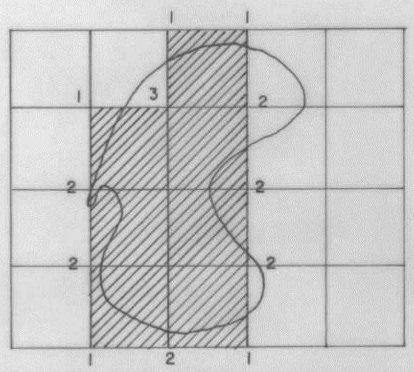
We have to collect these numbers travelling clockwise ↻ .

Observe that there are several strings of digits 1, 2 and 3 corresponding to the above chain, depending on the starting point:

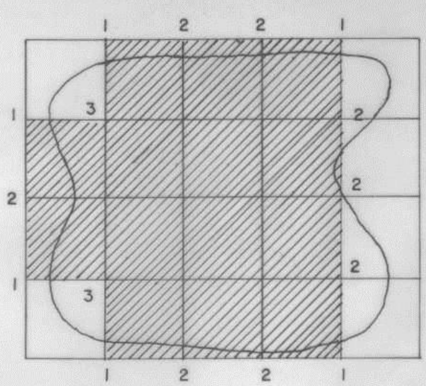
12131131213113	(A)
21311312131131	(B)
13113121311312	(C)
31131213113121	(D)
11312131131213	(E)
13121311312131	(F)
31213113121311	(G)
12131131213113	(H)
21311312131131	(I)
13113121311312	(J)
31131213113121	(K)
11312131131213	(L)



112331212122213113



112221212213



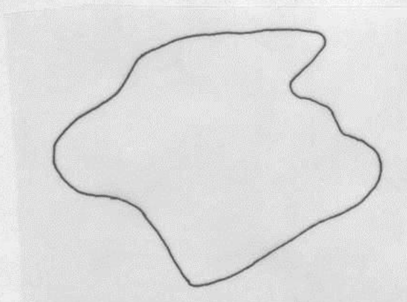
1213122122212213

TABLE 'EXAMPLES OF SHAPE NUMBERS'

13121311312131 (M)
31213113121311 (N)

Observe also that one of them is a minimum, when regarded as a number in base 3. (E) in the above example.

3. Select the chain that is minimum as the chain that represents the region. In the example, we conclude that the shape number of



is 1 1 3 1 2 1 3 1 1 3 1 2 1 3.

Observe that the minimum chain always starts with a 1, since in every closed curve there are at least four 1's (four salient corners)[Perceptrons: Minsky and Papert].

Other examples are found in the table 'EXAMPLES OF SHAPE NUMBERS.'

What size of grid? What orientation of such grid? Unless we give a procedure that normalizes these questions and provides unique answers, we will end up with a region having several shape numbers: see figure 'SEVERAL SHAPE NUMBERS' below.

The posture we adopt is that we will normalize (give a unique value for) the orientation of the grid, but its size (of the grid) will be a parameter that will allow us to vary the shape number of a region, to have less or more precision, as desired. Nevertheless, although the size of the cell of the grid varies according to the precision, the number of segments of the grid (sides of each cell) into which the region will be mapped is no longer at user's will, but it is dictated by the precision he specifies.

The orientation of the grid is not arbitrary, but it is made to coincide with the major axis of the region. The reason is clear: we want each region

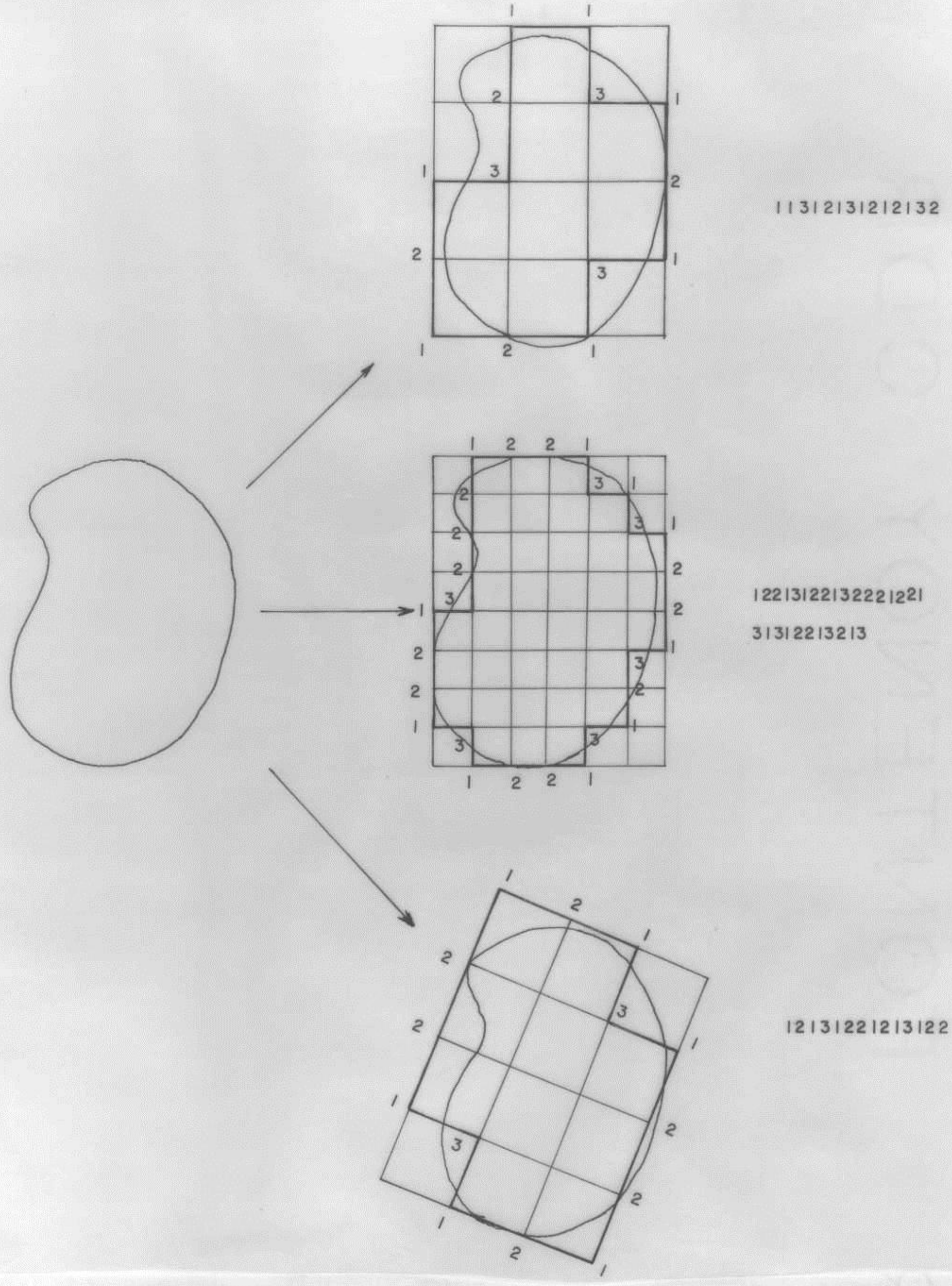
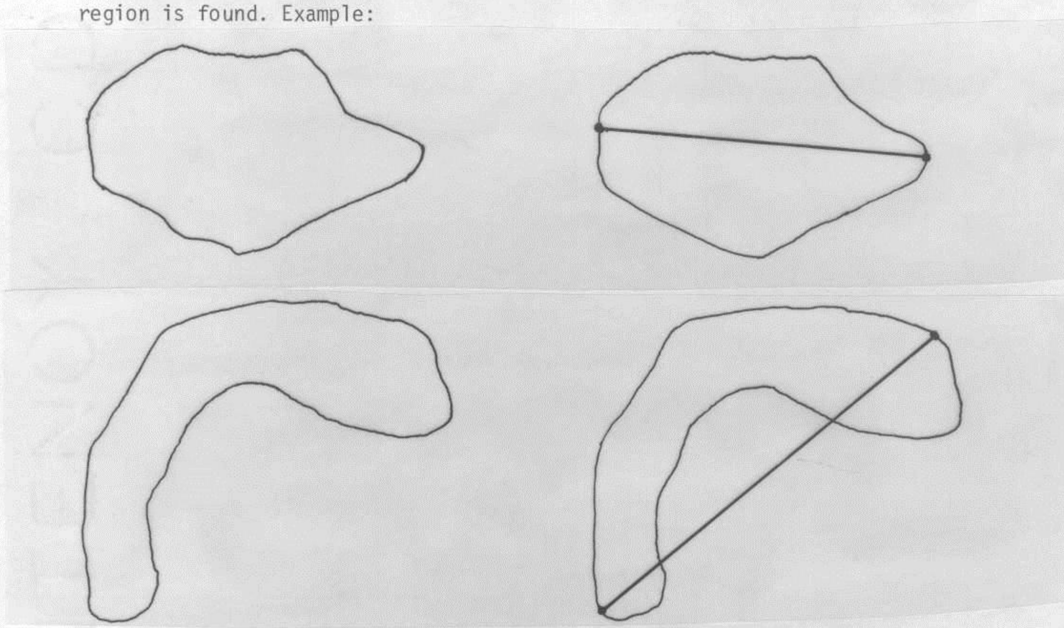


FIGURE 'SEVERAL SHAPE NUMBERS'

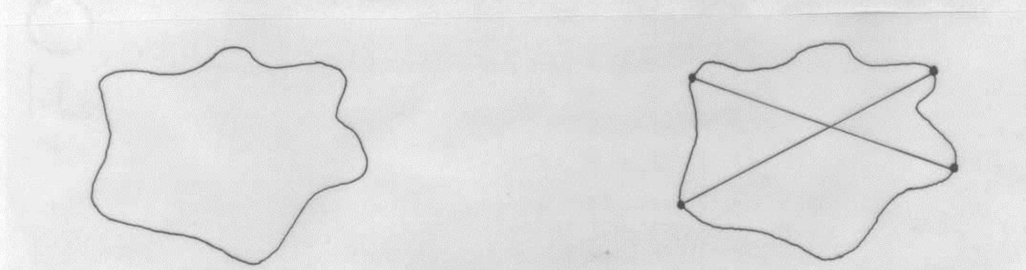
A region may give rise to several numbers that describe it. A canonical procedure is needed to avoid this. See text.

to carry along with it its own direction of the grid. In this manner, if you rotate the region, the grid rotates the same amount and a code is obtained invariant under rotations.

Procedure to achieve a unique orientation. First the major axis of the region is found. Example:

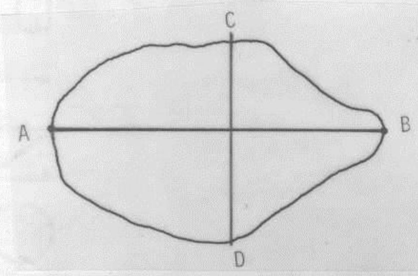


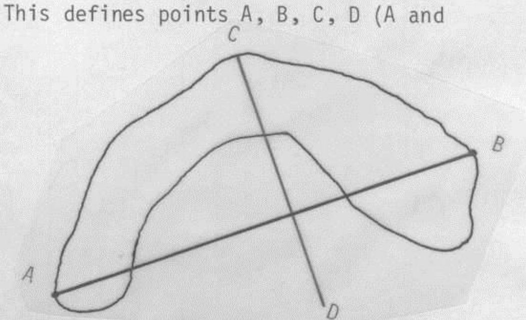
To do this, find the two points on the perimeter that are furthest apart. Ocassionally a region has two major axis:



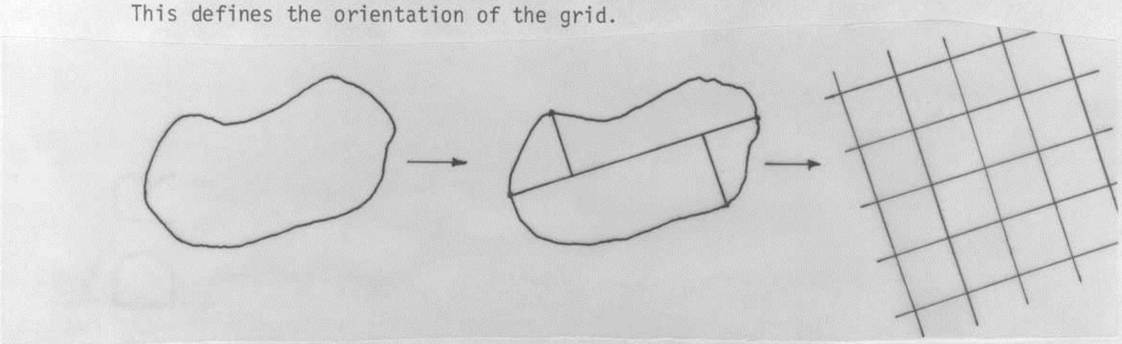
When this is the case, select one of them, according to certain criteria; for instance, that which produces the shortest minor axis.

Second, the minor axis is found. This defines points A, B, C, D (A and

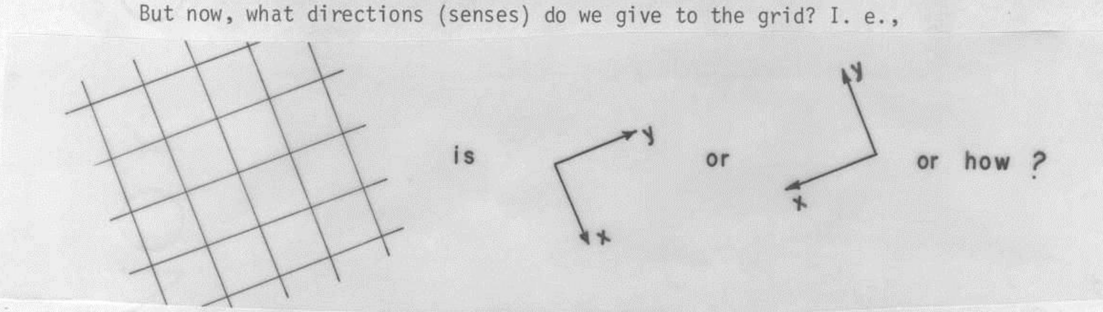




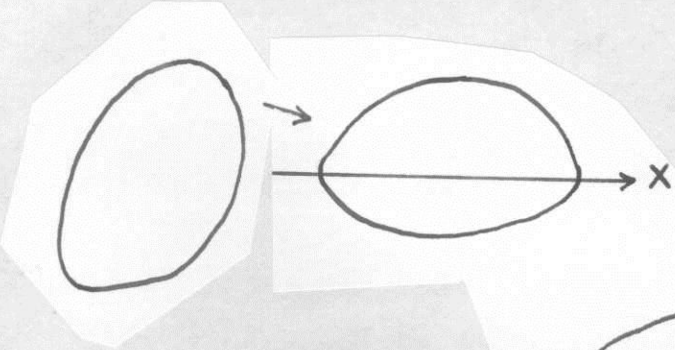
B are the extrema of the major axis; C and D are so for the minor axis) through which a rectangle (the basic rectangle of the region) can be drawn. This defines the orientation of the grid.



But now, what directions (senses) do we give to the grid? I. e.,

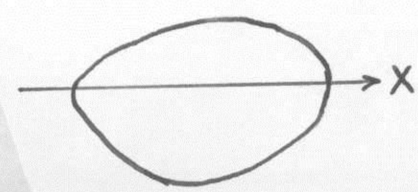


The rule we follow is to keep the major axis parallel to the X axis (horizontal) and as low as possible:

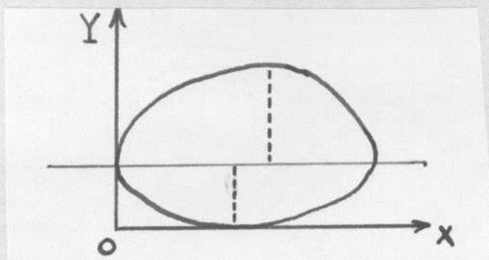


X axis parallel to the major axis

and not

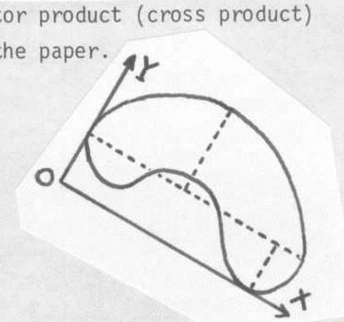


and therefore the Y axis is perpendicular to the X axis, and the Y axis goes positive (Y increases) in the direction of the longer segment of the minor axis (this axis is divided in two segments by the major axis):



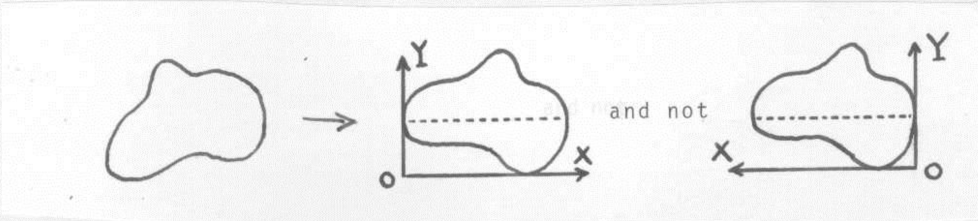
The axis X, Y are placed such that the vector product (cross product) X ⊗ Y points toward the reader, away from the paper.

The origin is placed at O; the lower left corner of the rectangle that encloses the curve.



To summarize, the rectangle that encloses the region has its longest side (X) parallel to the major axis. The shortest side of the rectangle is parallel to the Y axis, and Y increases away from the major axis and in the side (of the major axis) where the longer of the two pieces of the minor axis lies.

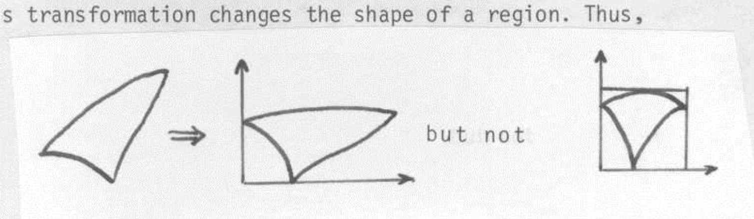
We have just found the basic rectangle (see definition) of the region. Notice that



because the mirror image of a region does not have the same shape as the original region (cf. definition).

It is convenient to make the X axis (major axis) horizontal and X increase to the right; Y is vertical and increases upwards, then. This is accomplished by a rotation of the figure in its plane, without need to flip it outside of its plane --no mirror images.--

We do not normalize X and Y such that the rectangle becomes a square; this transformation changes the shape of a region. Thus,



Excentricity. According to the definition of excentricity, we give several examples.

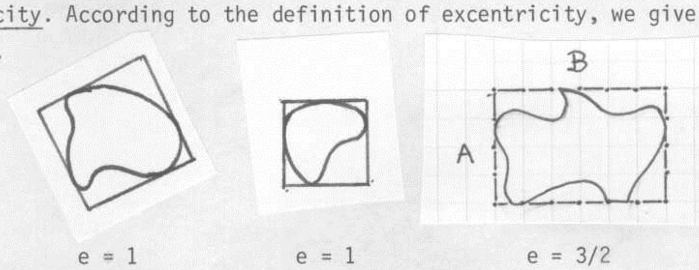
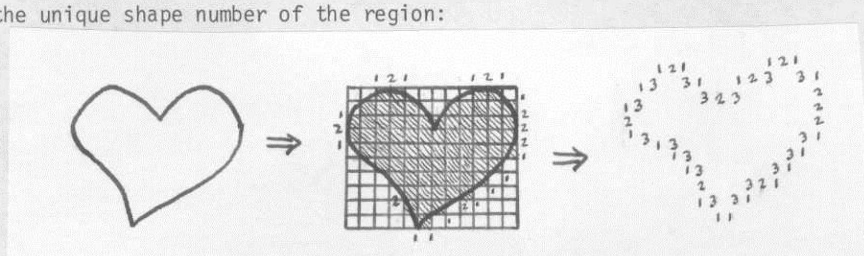


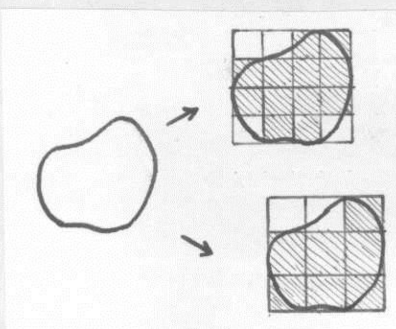
FIGURE 'EXCENTRICITIES'

Procedure to achieve a unique shape number. Given the basic rectangle with the region inside, we could then place a grid of a given (fixed) size, v. gr., 12 by 20 cells, on top of the rectangle, in order to extract the unique shape number of the region:



Nevertheless, we do not adopt this method. Instead, we allow the user to tell us how many digits he wants his shape number to contain. That is known as the order of the shape number.

The order of the shape number.(def) It is the number of ternary digits that the shape number contains. Example:



1212213121312123 order = 16

112212212123 order = 12

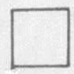
The order is always even, because the boundary is closed.

It is clear that the same shape gives rise to several shape numbers. But, given n, the shape number of order n of that shape is unique.

Shortly a procedure will be shown to find the shape number of order n of a region, for a given n. Before that, however, we present the families of discrete closed shapes of several orders.

<u>All the shapes of order 4</u>. These are all the regions that we can form with four sticks of the same size, provided we can place them only collinearly or at 90 degrees with respect to each other.

There is only one closed shape of order 4, the square:



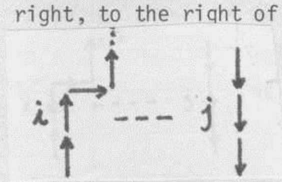
1 1 1 1

This is the most primitive or fundamental shape. Imagine you are looking at things very far away; you can not really differentiate much. All objects would look round (square, in this paper) and equal.

<u>All the shapes of order 5</u>. No shape number of odd order represents a closed figure. For a closed figure,

number of corners = number of sticks = order of figure.

But all closed figures have in front of a stick i going upwards, somewhere else (to its right, to the right of i) a corresponding stick j going

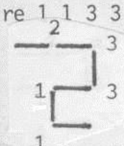


downwards. And similarly for horizontal sticks. They come in pairs. Therefore, the shape number is even for a closed curve. That is, for regions.

We may open the door to open curves (open shapes?) if we say

order of figure = number of corners; not necessarily equal to number of sticks.

In this case, figure 1 1 3 3 2 (order 5) is

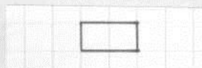


with six sticks

This paper does not deal with open figures.

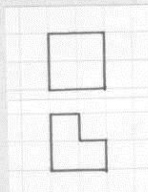
Not all ternary numbers with even number of digits are shape numbers. Most of them do not close.

All the shapes of order 6.



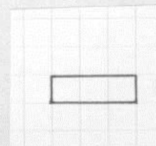
1 1 2 1 1 2

All the shapes of order 8.



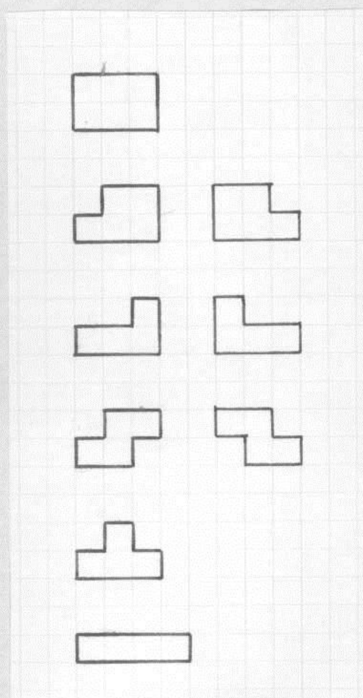
1 2 1 2 1 2 1 2

1 1 2 1 2 1 1 3

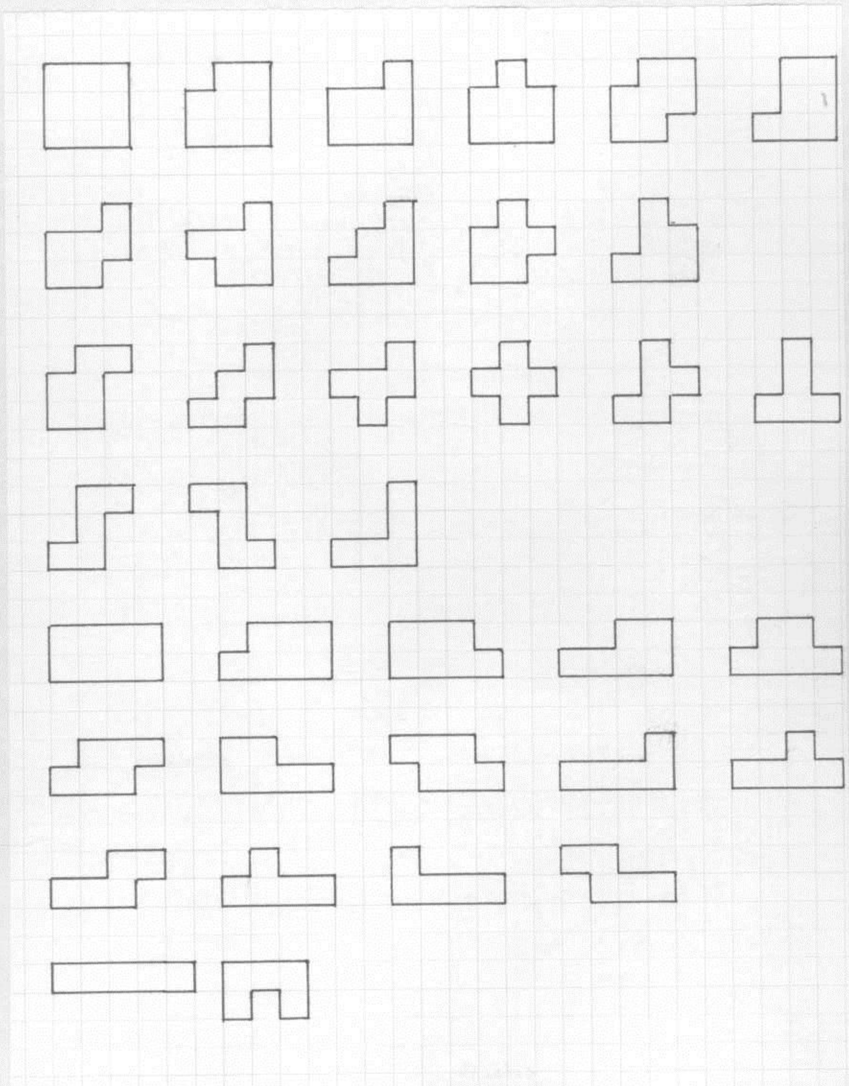


1 2 2 1 1 2 2

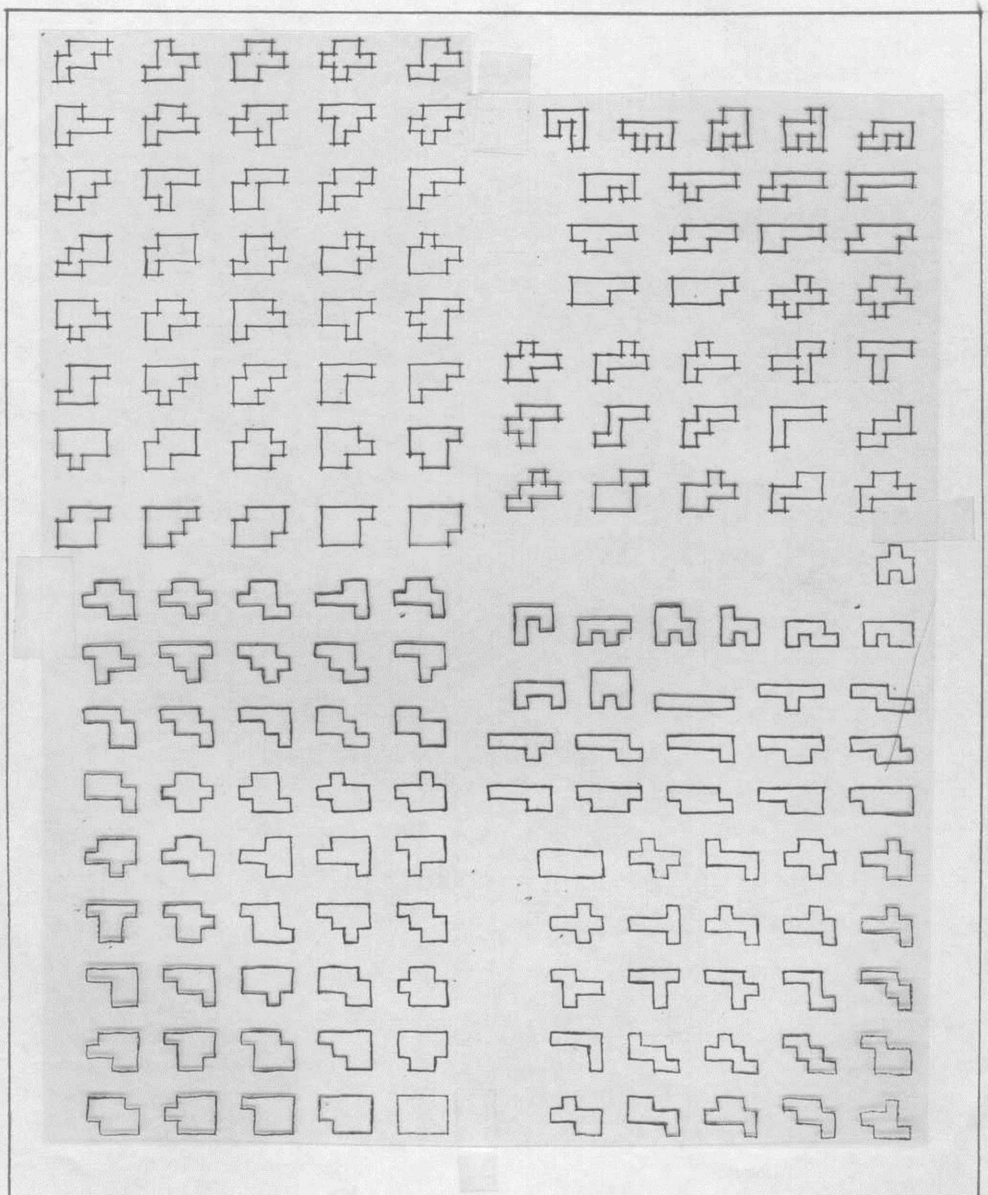
All the shapes of order 10.



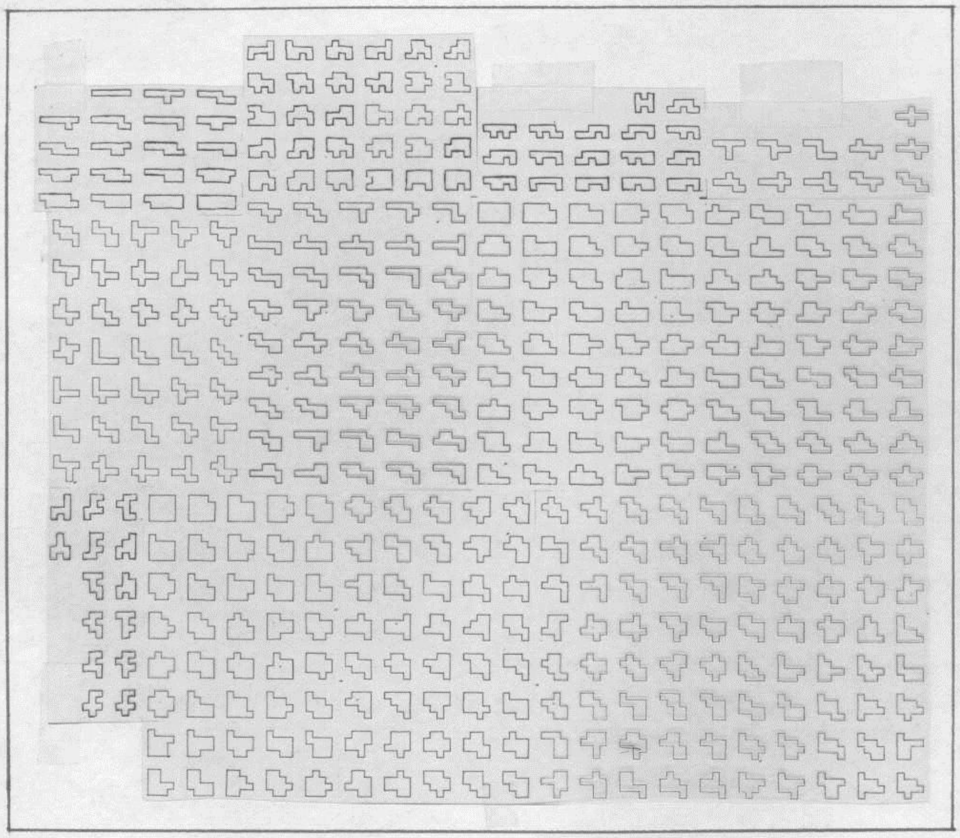
All the shapes of order 12.



All the shapes of order 14.



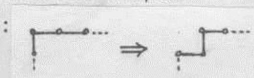
All the shapes of order 16.



We do not know a formula to compute the number of (closed) shapes of order n. There are 1 shapes of order 4

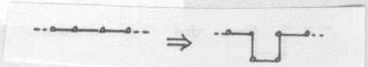
3	8
9	10
36	12
	14
	16

A procedure to find all the shape numbers of a given order

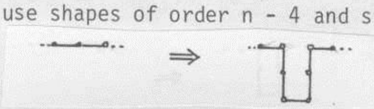
Suppose we want to find out all the shapes (or shape numbers) of order 12. We begin from the rectangles of order 12, and produce from them new shapes of order 12, through the procedure of "sinking the corners": 

Each time we produce a new shape, it becomes a candidate for sinking its corners.

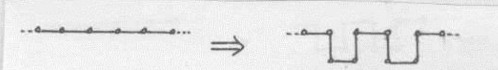
We then begin from all the shapes of order 10 (that is, n - 2) and produce new shapes of order 12, through the procedure of "sinking the edges":



This produces shapes with holes of depth 1.

If possible, use shapes of order n - 4 and sink their edges to obtain holes of depth 2: 

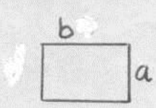
and also sink two holes of depth 1, simultaneously and in different parts of the chain:



Use in this manner until no longer possible, holes of depth 3, 4, etc.

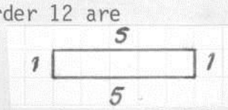
Each hole of depth k increases the order of the altered shape by 2k.

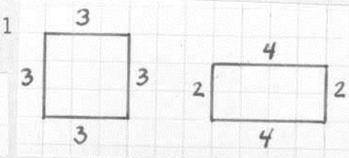
To find all the rectangles of order 12, observe that a + b = 12/2 = 6.



therefore the possible values for a and b are 3,3
4,2
5,1

and the only rectangles of order 12 are



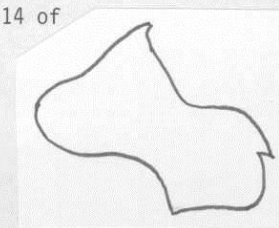


The obtention of all the shapes of order 12 illustrates fully the procedure. See Table 'OBTENTION OF ALL SHAPES OF ORDER TWELVE.'

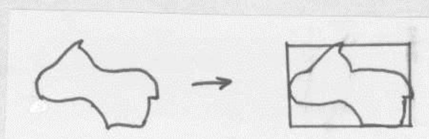
How to find the shape number of order n of a region.

The procedure is:

(1) Find the major and minor axis, and the basic rectangle of the region.
Example: find the shape number of order 14 of



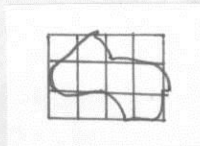
We proceed:



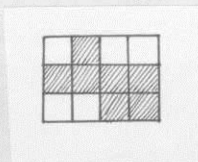
e = 4/3

(2) Find the rectangle of order n (cf. 'A procedure to find all the shape numbers of a given order') with excentricity closest to that of the region. This rectangle will be of size a,b such that 2(a+b) = n and $b/a \approx e$. In practice, it is better to aproximate the longer side of the rectangle instead of the excentricity. From the equations shown above, one can deduce that the longer side is b = (n/2)(e/1+e). Select a rectangle with longest side closest to that quantity.

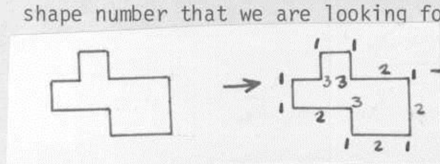
Lay this rectangle so as to cover the region, and make a grid of a by b square cells:



(3) Make black (=1) all those cells falling 50% or more inside the figure; leave white (=0) all others.



The boundary of this black region, expressed in the 1-2-3 notation, is the shape number that we are looking for:



13113212121321

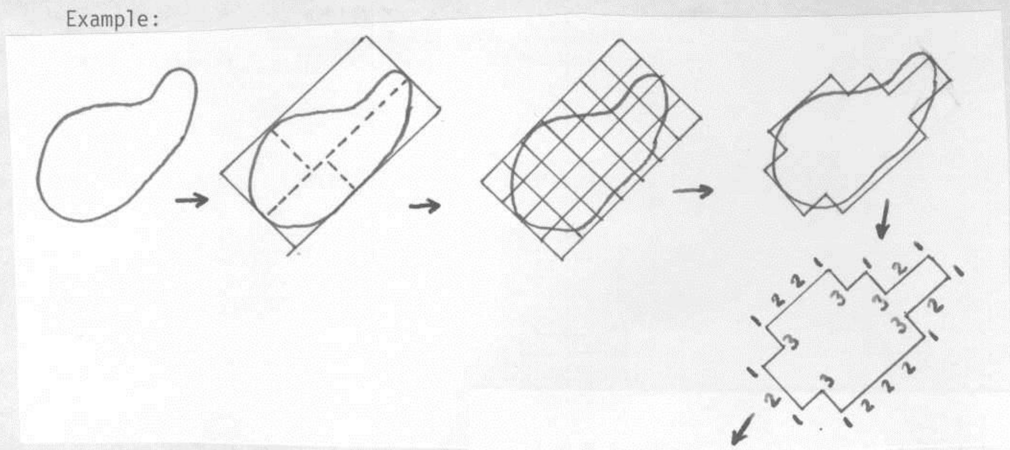
↓

11321212132112

(This procedure has been explained in more detail under § 'The Shape numbers').

Remember to write down the digits of the chain travelling it clockwise, and selecting the starting point in the 1 that makes the chain number the smallest of the n different chain numbers.

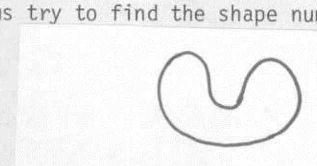
Example:



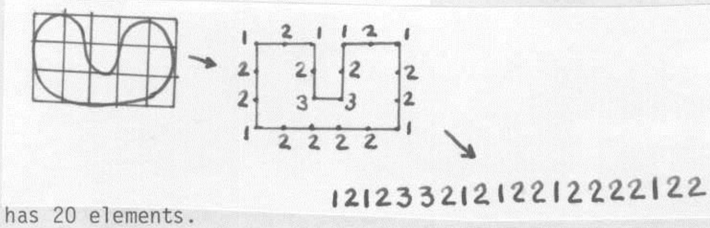
1 1 2 3 1 2 2 2 1 3 1 2 1 3 1 2 2 1 3 1 3 2

Notice that the resulting shape number is indeed of order n.

This will not be true if the figure has depressions ("holes") in its boundary. Let us try to find the shape number of order 16 of



Then,



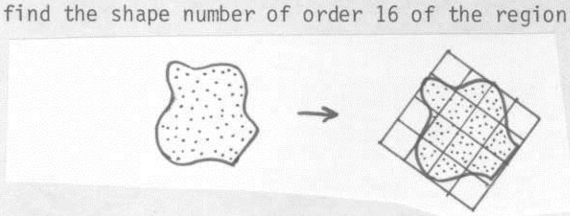
which has 20 elements.

The depression in the boundary makes the order bigger. We already had discovered this phenomenon: "each hole of depth k increases the order of the shape by 2k." When this happens, i. e., we look for a shape of order n and find one of order n+2d, try next to look for a shape number of order n-2d. We know that, because of the presence of the holes, the shape number n-2d will be increased by an amount equal to the "hole excess" 2d, thus yielding the desired order n. This relation holds only approximately, since the size of the holes of order n is smaller than those of order n-2d. Thus, in practice, we will have to try several orders to start with, namely n-2d, n-2d+2, n-2d+4, ..., n-2, and when we obtain a shape number of order n, that is it.

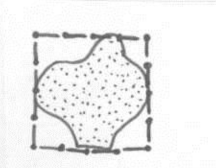
Alternative procedure for finding the shape number of order n. An example illustrates this variant.

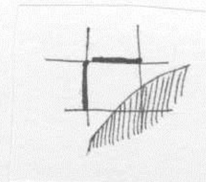
Given the region, find its major and minor axis, as well as the basic rectangle, as before.

Example: find the shape number of order 16 of the region below.

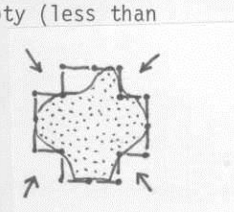


Now place n sticks (n is 16 in the example) of equal size on the rectangle:

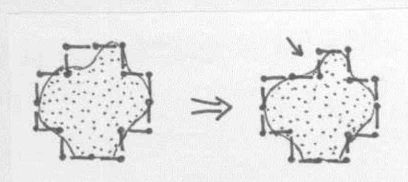




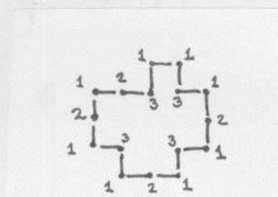
Now begin to push corners: if two sticks surround a cell nearly empty (less than 50%), push that corner:



and we continue the process until no further progress is possible.



The final chain is



which is now travelled clockwise, giving 1 3 1 2 1 3 1 2 1 3 1 2 1 2 3 1 which is now rotated circularly (shifted around) until we reach the smallest value for the whole chain:

1 1 3 1 2 1 3 1 2 1 3 1 2 1 2 3

This is the shape number of order 16.

Do not forget to push through holes, too, if needed.

If the resulting shape number is of order larger than n, we proceed as already explained.

Properties of the shape number. Insensitive to orientation of the region. Insensitive to position of the region. Insensitive to size of the region. Insensitive to origin of the chain.

It is therefore appropriate to think that the shape number of a figure indeed describes its shape (q.v.).

Also, since it is possible to compute the shape number of a region without reference to a table of stored shapes (canonical shapes), we avoid making correlations or comparisons of shapes. That is, the shape number of a region can be deduced solely from the region.

In addition, we can vary the precision of the resulting shape number. This is done with the order of the shape number, that is, the size of the sticks that we use to find that shape number.

The next section shows how to measure quantitatively the difference in shape of two regions denoted by their shape numbers.

MEASUREMENT OF THE SIMILARITY OF TWO SHAPES

(Def) Two regions a and b have the same shape of order n if and only if the shape number of order n of a is equal to the shape number of order n of b.

(Def) Two regions have identical shape if for all n, they have the same shape of order n.

That is, if for all n, the shape number of order n of a coincides with the shape number of order n of b.

Now, notice the following:

(1) All regions have the same shape of order 4, since there is only one shape of order 4, the square 1 1 1 1.

(2) If two regions do not have identical shape, there is a minimum k at which the shapes are not the same, that is, the shape number of order k of a is different from the shape number of order k of b.

(3) Those two figures will have the same shape of order n for n less than k; those two regions will have different shapes of order n for n greater or equal to k.

That is, any shape number of order smaller than k of one region will be the same for the other region (when computed at the same order); all the shape numbers of order greater or equal than k will be different.

That is, any shape number of order smaller than k of one region will be the same for the other region (when computed at the same order); all the shape numbers of order greater or equal than k will be different.

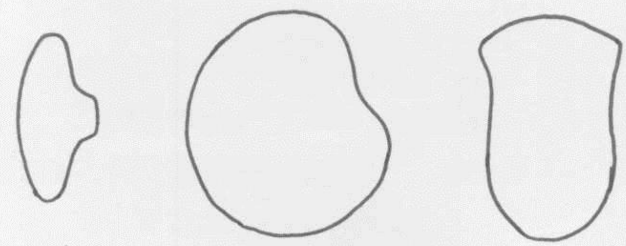
(Def) The maximum order at which the shapes of two regions agree (at which the two regions have the same shape) defines their degree of similarity. Using the k defined in (2) above, the degree of similarity between a and b is k-2.

Informally speaking, it is the maximum precision (resolution, size of the magnifying lens) that still confuses two shapes, by reporting the same shape number for both. The smaller the degree of similarity between two regions, the more different in shape they are.

Any region is similar to any other at degree 4.

Two figures with identical shape have a degree of similarity equal to infinity.

Example. Let us examine the regions



A B C

Then,

shape numbers of order 4 are	$S_4(a)$= 1 1 1 1	$S_4(b)$= 1 1 1 1	$S_4(c)$= 1 1 1 1
of order 6	$S_6(a)$= 1 1 2 1 1 2	S_6b)= 1 1 2 1 1 2	$S_6(c)$= 1 1 2 1 1 2
of order 8	$S_8(a)$= 11221122	$S_8(b)$=12121212	$S_8(c)$= 12121212
of order 10	$S_{10}(a)$=1122211222	$S_{10}(b)$= 1131212122	$S_{10}(c)$= 1212212122
of order 12	$S_{12}(a)$=112221131213	$S_{12}(b)$=121221221213	$S_{12}(c)$=121222121222
of order 14	$S_{14}(a)$=11222211231132	$S_{14}(b)$=12121312212123	$S_{14}(c)$=113(122)^{3}13

Thus, we conclude

that the degree of similarity between a and b is 6, written $a \approx_6 b$.
that the degree of similarity between a and c is 6, written $a \approx_6 c$.
that the degree of similarity between b and c is 8, written $b \approx_8 c$.

Example. The shape numbers of figures D, E and F (shown in next page) are:

	D	E	F
order 8	12121212	12121212	12121212
order 10	1121221123	1131212122	1212212122
order 12	112131131123	113113113113	121222121222
Order 14	11232121221222	11231131131223	12122221212222

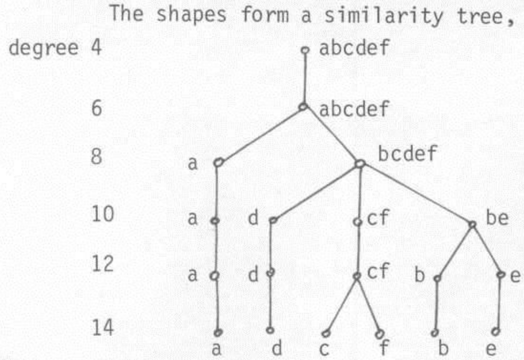
Therefore, the degree of similarity of d and e is 8;
the degree of similarity of e and f is 8;
the degree of similarity of d and f is 8.

We can also compare these figures against the previous three regions; and concentrate the degrees of similarity in the following similarity matrix:

	A	B	C	D	E	F
A	∞	6	6	6	6	6
B	6	∞	8	8	10	8
C	6	8	∞	8	8	12
D	6	8	8	∞	8	8
E	6	10	8	8	∞	8
F	6	8	12	8	8	∞

The similarity matrix is symmetrical.

The shapes form a similarity tree, as follows:



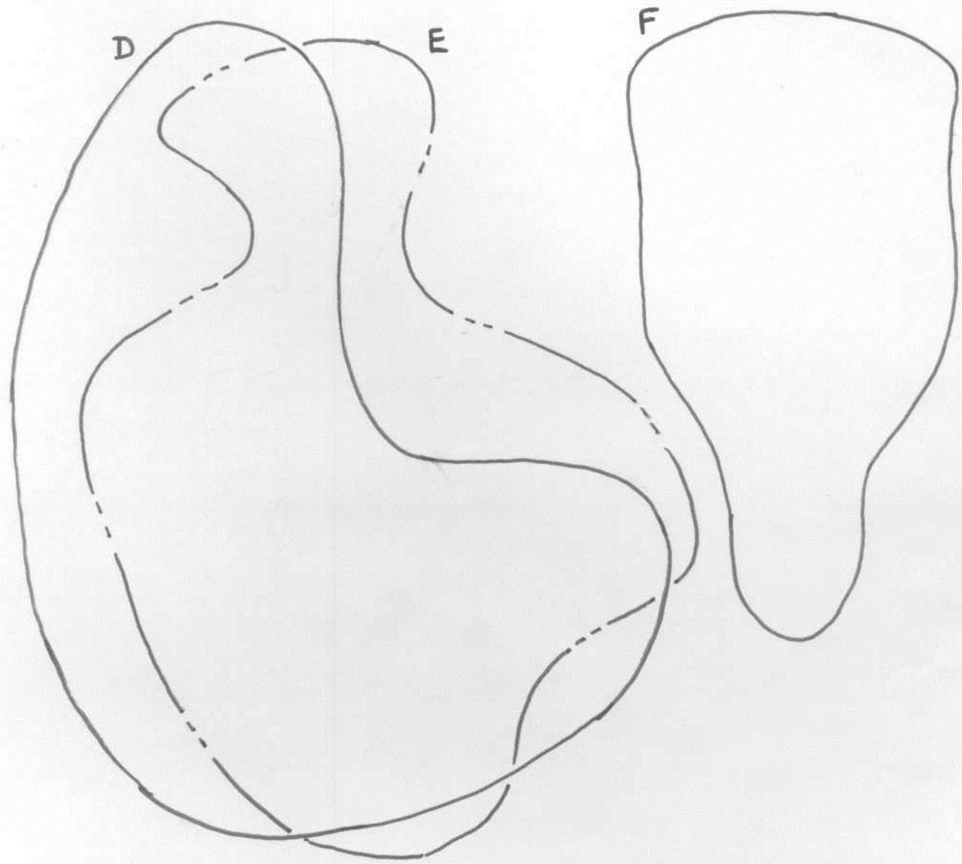


FIGURE 'SIMILARITY BETWEEN REGIONS'

The regions shown here are compared among themselves and with regions a, b and c shown before. The results can be expressed in a similarity matrix and in a similarity tree; they are sown in the previous page.

If the degree of similarity of f1 and f2 is 12, and that of f3 and f4 is 24, we can not conclude that f3 and f4 are "twice as close in shape" as f1 and f2. This is like the temperature: a body at 100°C is not twice as hot as one at 50°C (if you do not believe it, convert them to degrees Farenheit, or to °K).

Equivalence relations for shapes. The relation "f1 and f2 have degree k of similarity (for a fixed k)" is not an equivalence relation:

(1) f1 has not a degree of similarity equal to k with f1. It is infinity.

(2)The relation is symmetrical.

(3) $f1 \approx_k f_2$ and $f2 \approx_k f3$ does not always implies $f1 \approx_k f3$. For instance, if f1 and f3 are very close in shape (degree 200 of similarity), we may have, when we compare each of them with a less similar f2,

$$f1 \approx_{30} f2,$$

$$f2 \approx_{30} f3,$$

but $f1 \approx_{200} f3$, instead of $f1 \approx_{30} f3$.

The relation "f1 and f2 have a degree of similarity of at least k (for fixed k)", is an equivalence relation.

(1) f1 and f1 have a degree infinity of similarity (thus, at least k).

(2) It is a symmetric relation.

(3) If f1 and f2 have a degree of similarity of at least k, then their shape numbers of order k are equal, $s_k(f1)=s_k(f2)$. If f2 and f3 also have a degree of similarity of at least k, we also have $s_k(f2)=s_k(f3)$. Therefore, $s_k(f1)=s_k(f3)$, which says that f1 and f3 have a degree of similarity of at least k.

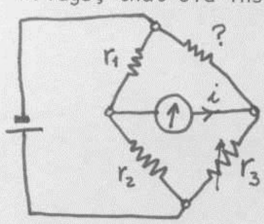
Thus, for instance, there are three equivalence classes for figures having a degree of similarity of at least 8: a representative for each class is found in § 'All the shapes of order 8'.

Remarks on the degree of similarity

No parsing is necessary. To find the degree of similarity between a and b, their shape numbers are compared for equality. Two shape numbers of different orders are inconmensurable. Two shape numbers of the same order are either equal or different. If different, that is it. There is no need to compare "how close in shape they are." String matching [Guzmán 66] is not needed.

To find out the degree of similarity, a binary search is used (or a modified binary search, if it costs more to compare --or deduce-- numbers at large orders than those at low orders): First see whether the shape numbers at order 8 are equal or not. Then compare the shape numbers at the highest required accuracy (say, 100). Then at the middle. Then at the middle of the remaining valid half. And so on.

Precision is not needed when comparing shape numbers. Think of the Wheatstone bridge, that old instrument used to measure the value of resistences. An



amperimeter says whether current i is zero or not. This amperimeter does not measure the resistance itself; it only says: current is 0. Stop! Then the value of the resistance is obtained by a formula that does not involve the current (since it is zero!). Naturally, it does not need to be a high precision amperimeter.

In our case, the degree of similarity is not measured or given by the shape comparison test. It is given by a process that uses the comparison test. If not yet satisfied, this process 'orders' the comparison test to compare a different pair of numbers, just asking from it (from the test) a binary decision.

Ultradistance. If we define the distance between two shapes a and b to be the inverse of their degree of similarity, then we could easily prove that it is not only a distance, but also an ultradistance. That is, it obeys $d(a,c) \leq Sup(d(a,b), d(b,c))$ in addition to the less demanding condition $d(a,c) \leq d(a,b) + d(b,c)$.

Distance. (def) The distance between two shapes a and b is defined to be the inverse of their degree of similarity. $d(a,b) \triangleq 1/k$.

Then d is an ultradistance, obeying

$d(a,a)=0$ (1)

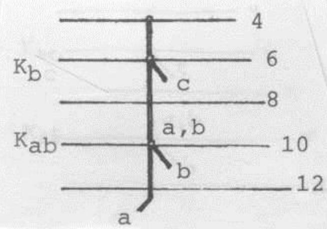
$d(a,b) \geq 0$; $d(a,b)=0$ if and only if $a = b$ (2)

$d(a,c) \leq Sup(d(a,b), d(b,c))$ (3)

Proof. (1) and (2) are obtained directly from the definition.
To prove (3), call $k_{ab} = 1/d(a,b)$ and $k_{bc} = 1/d(b,c)$ the degrees of similarity and distances between a--b and b--c.

Then k_{ab} is either greater than, equal to or less than k_{bc}.

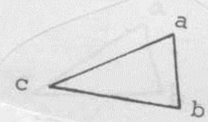
Case 1. $k_{ab} > k_{bc}$. Then in the tree, k_{bc} is above k_{ab}. The branching of c occurs nearer to the root than the branching of a.



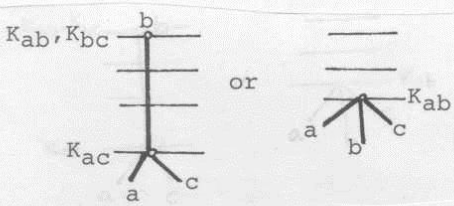
Then $k_{ca} = k_{bc}$ also.

And $d(a,c)=1/k_{ca} = 1/k_{bc}=\sup(1/k_{bc},1/k_{ab})$

$= \sup(d(b,c), d(a,b))$ since $k_{ab}>k_{bc}$.



Case 2. $k_{ab} = k_{bc}$.

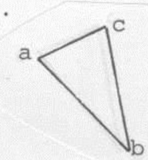


The branching of c from b occurs at level k_{ac} or above.

Then $k_{ca} \geq k_{bc}$ and

$d(a,c) = 1/k_{ca} \leq 1/k_{bc} = \text{Sup}(1/k_{bc}, 1/k_{ab})$

since $k_{bc} = k_{ab}$.



Case 3. $k_{ab} < k_{bc}$.

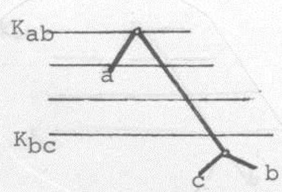
In this last case k_{bc} is below k_{ab}. Branching of c from b occurs below branching of a from b.

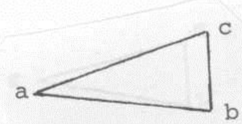
Then $k_{ca} = k_{ab}$ also

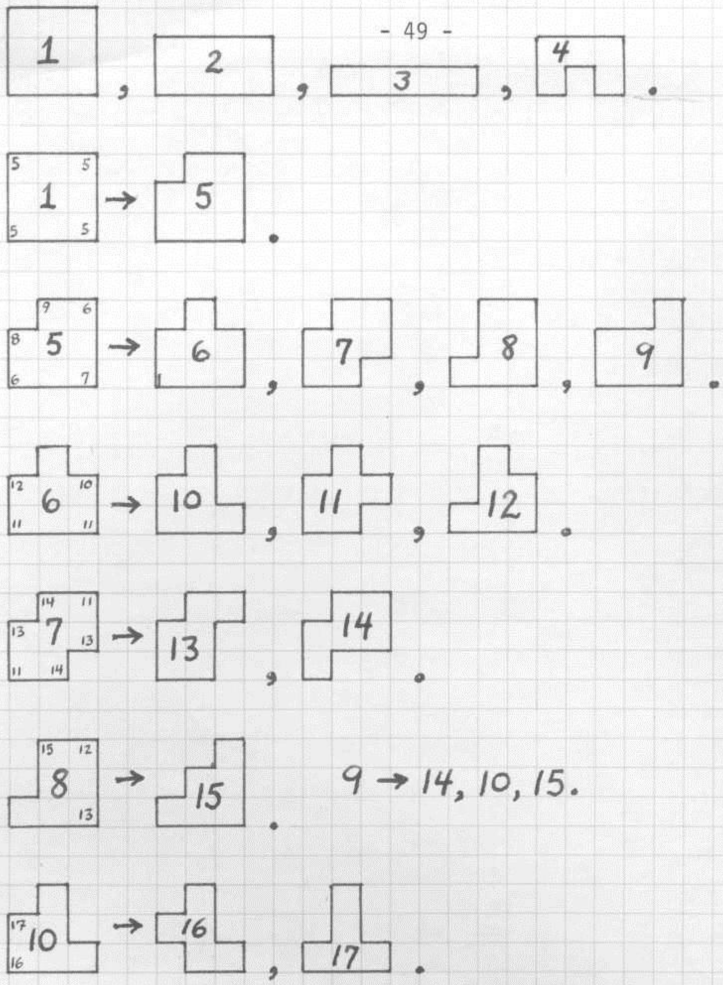
and $d(a,c) = 1/k_{ca} = 1/k_{ab}$

$= \text{Sup}(1/k_{bc}, 1/k_{ab})$

$= \text{Sup}(d(b,c), d(a,b))$ since $k_{bc} > k_{ab}$.







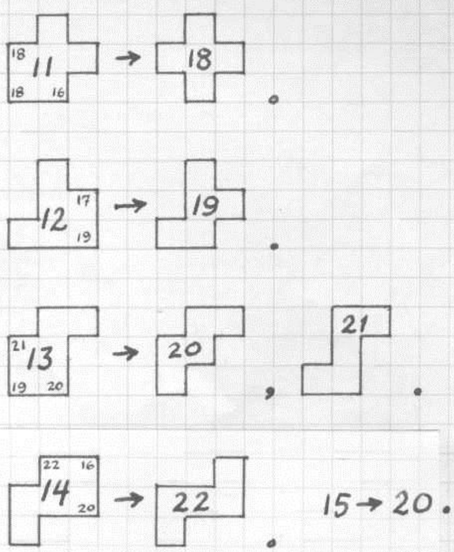


TABLE 'OBTENTION OF ALL THE SHAPES OF ORDER 12'
Part 1 of 2.

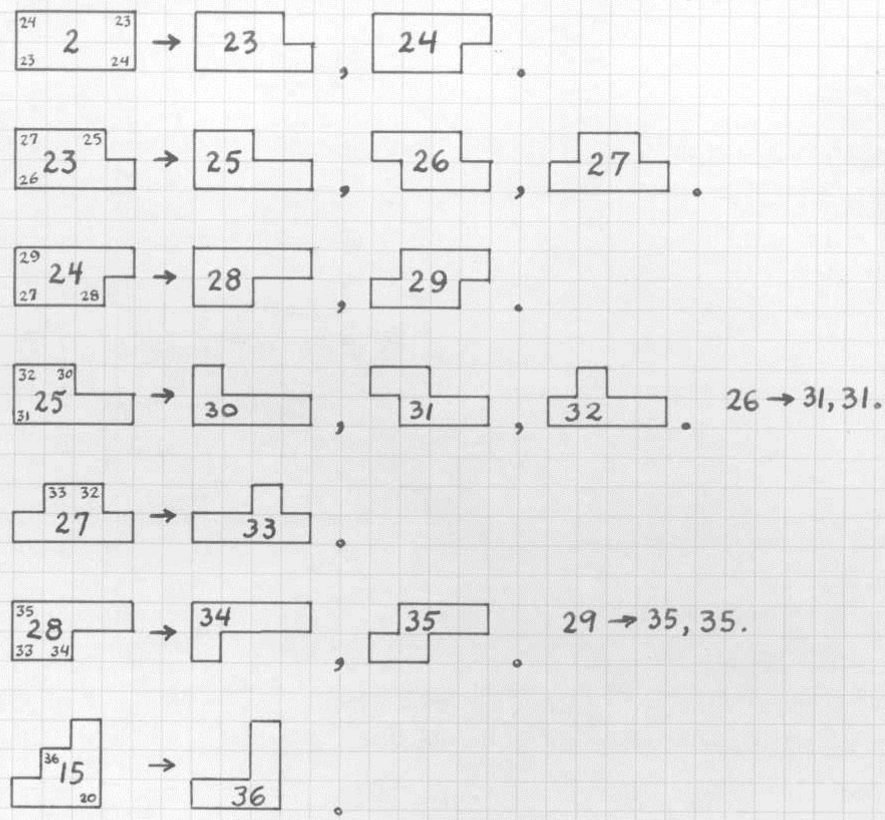


TABLE 'OBTENTION OF ALL THE SHAPE NUMBERS OF ORDER 12'
Part 2 of 2.

Any family of shapes is produced from the basic rectangles by two procedures, explained in the text: "sinking the corners" and "sinking the sides."

Comments on this theory of shapes

Shape numbers are not invariant under (1) reflexions (mirror images), (2) skewing, (3) unequal expansion along the axis X and Y.

These transformations (1)-(3) alter what could be considered the (intuitive) shape of a figure. At the end of the paper a Theory "B" of shapes is presented, where condition (3) is violated, i.e., a circle and an ellipse have the same Bshape number.

Problems with this theory of shapes

1. Occassional loop in the similarity tree. Due to noise or the 50% requirement for quantization, and at low orders, sometimes it is observed a transitory divergence and then convergence in the shapes of two regions,

v. gr., $s_8(a) = s_8(b)$

$s_{10}(a) \neq s_{10}(b)$

$s_{12}(a) = s_{12}(b)$

$s_{14}(a) \neq s_{14}(b)$

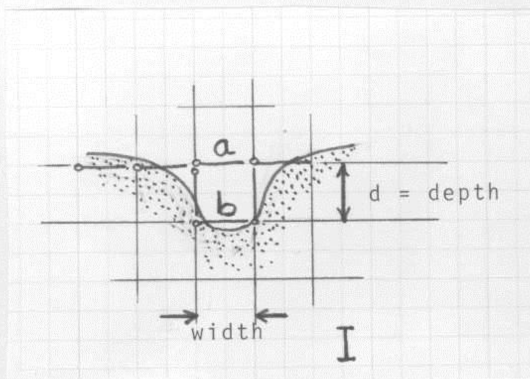
$s_{16}(a) \neq s_{16}(b)$

$\vdots$

i.e., they were already different at order 10, but they are again equal at order 12 (however, only to separate soon forever). This still gives a unique number for the shape of a region, but makes the definition of degree of similarity less attractive, and the procedure to find it, unreliable. Only loops of size 2 (such as the example given) have been found, infrequently. These loops disappear if you elliminate half of the orders (cf. Suggestion 8b).

2. Non existent shape numbers. Shape number of order o may ocassionally not exist for a given figure, due for instance to symmetrical holes of type I in figure 'HOLES AND DEGENERATE SHAPES'. This does not bother the similarity procedure, but it is a nuisance not to have that shape number. See also Suggestion 8a.

3. Quantization of the excentricity. The basic rectangles of order 12 have



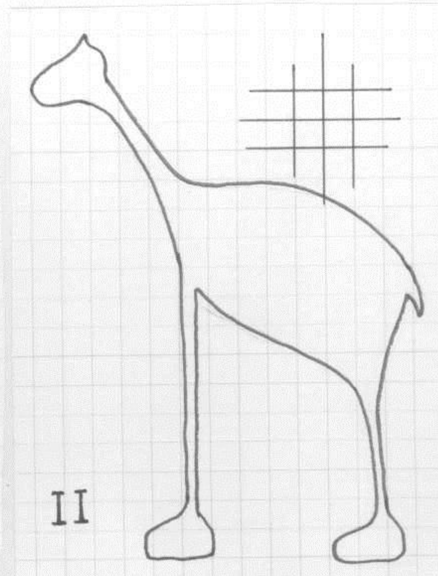


FIGURE 'HOLES AND DEGENERATE SHAPES'

I. A depression of depth d increases the shape number by 2d.

II. Degenerate regions split the discrete shape but do not have a shape number.

excentricities equal to 1 (the square of 3 by 3), 2 (the rectangle of 4 by 2), and 5 (the rectangle of 5 by 1). For an object of excentricity 1.6, one of these has to be used. An error is going to be commited in any case. There seems to be no way out of this. See Suggestion 5.

We now present a theory that has none of these problems.

THEORY "B" FOR SHAPE DESCRIPTION AND SHAPE COMPARISON

To obtain this new theory, we will make some changes to the current theory:

1. Force the excentricity of any region to be equal to one, by performing an unequal dilation of its axis, $X' = a\,x$, $Y' = b\,y$, $a \gtrless b$.
   The only discrete Bshapes that now exist are those obtained from squares. All the rectangles have disappeared.

2. Do not go into depressions (part I of figure 'HOLES AND DEGENERATE SHAPES') with width smaller than the size of the side of the cell of the grid. This avoids degenerate shapes.
   That is, if a region is "scratched" by thin lines (thiner than the size of the grid) that belong to the background, we either ignore them (act as if they were not there) or else, if they can not be ignored, this theory "B" says that the size of the grid is inappropriate to describe such region, and that its Bshape does not exist at this order. Higher resolution is needed.

3. Let the depressions where the sticks do go in (because they are wider than part I of figure 'HOLES AND DEGENERATE SHAPES') generate Bshape numbers having a number of (ternary) digits longer than the expected order. That is, do not correct the anomaly that these depressions cause. The perimeter of the Bshapes does not tell anymore its order.

4. Elliminate the orders that are not powers of two. The only valid orders for Bshape numbers are 4, 8, 16, 32, ... These numbers still indicate the number of sticks to place around the basic square of the region

(cf. § 'Alternative procedure to find the shape number of order n'). The procedure is the following.

How to find the Bshape number of order n

1. Find the basic rectangle of the region and convert it to a square. Declare that the Bshape number does not exist if the region has necks (isthmus) or depressions (channels, fjords) narrower than 2^{2-n} or 4/o.

2. Make a grid by dividing the side of the basic square into o/4 segments.

3. Mark with a 1 each cell of the grid of step 2 that is more than 50% contained in the region (you could also go through the variant described in § 'Alternative procedure for finding the shape number of order n'). The collection of grid squares containing a 1 form a discrete Bshape.
4. Find the shape number of the discrete Bshape of step 3, and give that as answer (even if it has more than o ternary digits).

The order o of a Bshape number is four times the number of parts into which the side of the basic square was divided. It is also the perimeter (measured by the number of sticks) of the basic square.

It is no longer the perimeter of the discrete Bshape; nor the number of ternary digits of the Bshape number.

An example of Bshape numbers is given in figure 'SOME BSHAPE NUMBERS'. Also, in the next few pages we show the Bshape numbers of order 4 to 64 for figures A to F.

The degree of similarity between the Bshapes of two regions is obtained as before. Definition unchanged.

Some examples are given in the following pages, for figures A to F.

Downwards constructability. Given the Bshape number of order o of a region, the Bshape number of order o/2 can be deduced from it, by regroupping appropriate sets of 4 neighboring cells into a cell for the lower order. Therefore, if two regions have the same Bshape number of order o, they will continue to have equal Bshape numbers of smaller order, until they cease to exist. This gets rid of problem 1, 'Ocassional loops in the similarity tree' of the former theory.

Upwards existence. If the Bshape number of order o of a region exists, the existence of numbers for higher orders is guaranteed: (1) the inexistence of channels or narrow parts of the region thiner than 4/o implies the inexistence of those narrower than 4/(o+i) for $i > 0$; and (2) wider depressions (wider than part I of figure 'HOLES AND DEGENERATE SHAPES') will produce valid parts of the Bshape number, although its number of digits may increase. This defeats problem 2 of the former theory, "non existent shape numbers."

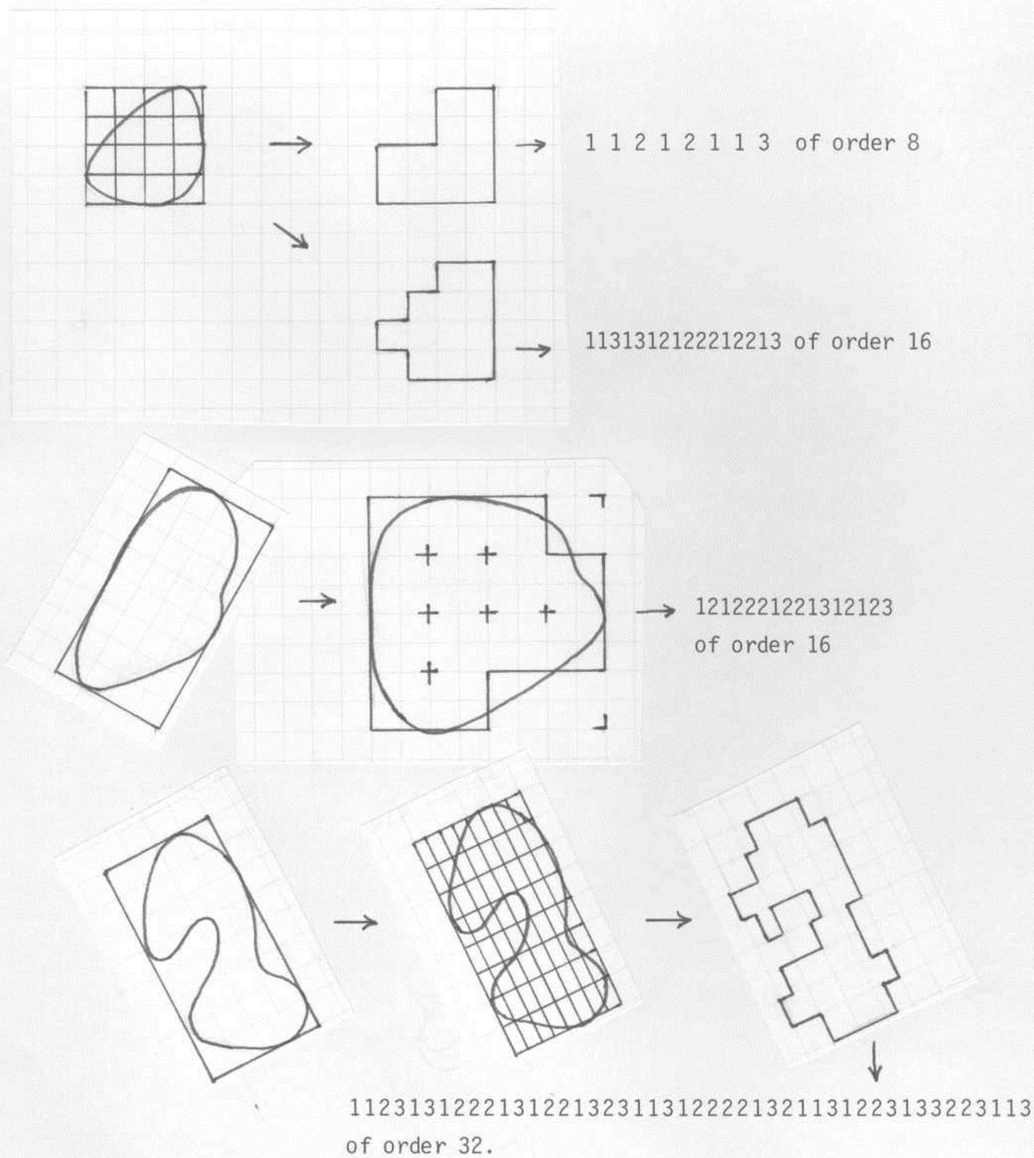


FIGURE 'SOME BSHAPE NUMBERS'

Finall y, problem 3 of the former theory "quantization of the excentricity" is not present in Theory "B" because all excentricities are now equal to 1.

Some examples of similarity comparison using theory "B" are given in table 'SIMILARITY TREE FOR THE BSHAPES OF REGIONS A TO F'.

Disadvantage of Theory "B". Squeezing along one axis is now a valid (Bshape preserving) transformation. Thus, either your application does not care for the excentricity or aspect ratio, or you carry it as another parameter, in addition to the Bshape number. I suppose you are going to be carrying other parameters of the region (length, orientation) anyway.

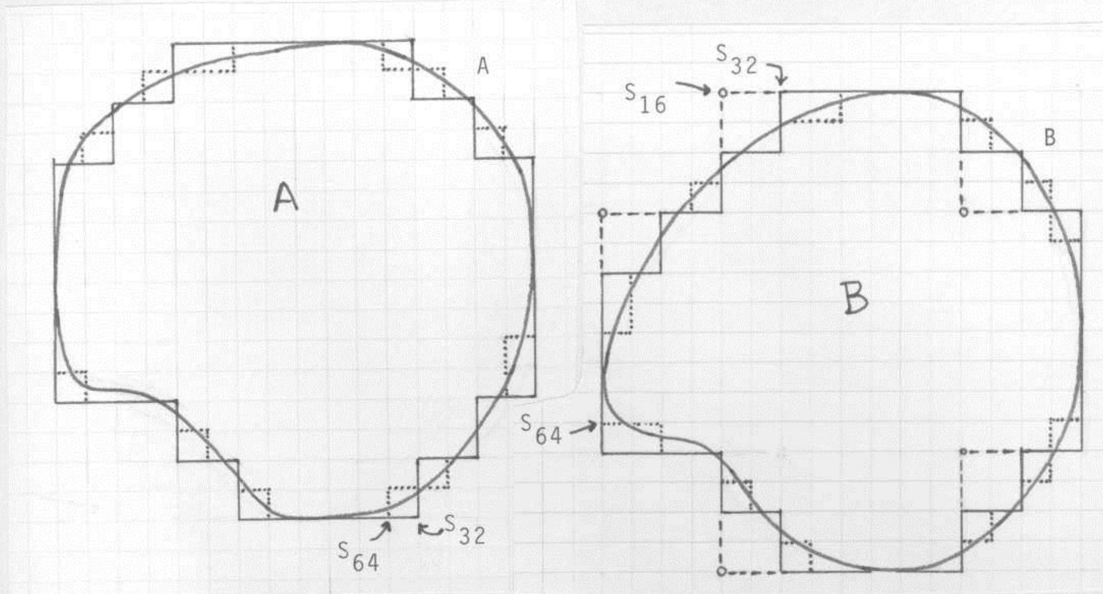
Also, more care needs to be exercised now when selecting the major and minor axis, to avoid noise perturbations (cf. suggestion 7).

SUGGESTIONS AND RECOMMENDATIONS FOR FURTHER WORK

1. Use other tessellations (triangles, hexagons) instead of the square grid. I would like to see the triangle and circle as primitive shapes at low orders.
2. Use eight directions for the sticks, not 4. This will produce more shape numbers of a given order, thus making the tables of canonical shapes larger. But this is safe because the deduction of the shape number does not involve table lookup or comparison with these canonical shapes.
3. Apply these theories to scene analysis of coloring books [Guzmán 71]; chromosomes; silouettes of industrial parts on a conveyor belt; hand printed digits and zip codes; automatic taxonomy of shapes of shoes, airplanes; insects (their outline) ; texture description for binary images.
4. Extend these theories to shapes with holes inside them.

5.A. (Refer to problem 3 of first theory). Distort slightly the basic rectangle of the region, together with the region, so as to have it coincide exactly with the rectangle chosen among the discrete shapes. the grid is now of rectangles that are almost squares.

5.B. (Refer to problem 3 of first theory). A better way to select, among the rectangles of order n and certain excentricity, is to minimize the discrepancy between the areas of the region and of the rectangle.



$S_4(a) = 1^4$ $\quad S_8(a) = (12)^4$ $\quad S_{16}(a) = (1213)^4$

$S_{32}(a) = 122(13)^2 212^3 (13)^2 12^3 (13)^2 12^3 (13)^2$

$S_{64}(a) = 12^3 (13)^4 221312^6 (13)^3 122312^4 132(13)^3 12^5 (132)^2 13123$

$S_4(b) = 1^4$

$S_8(b) = (12)^4$ $\quad S_{16}(b) = (1213)^4$

$S_{32}(b) = 122(13)^2 12^3 (13)^2 122(13)^2 2122(13)^3$

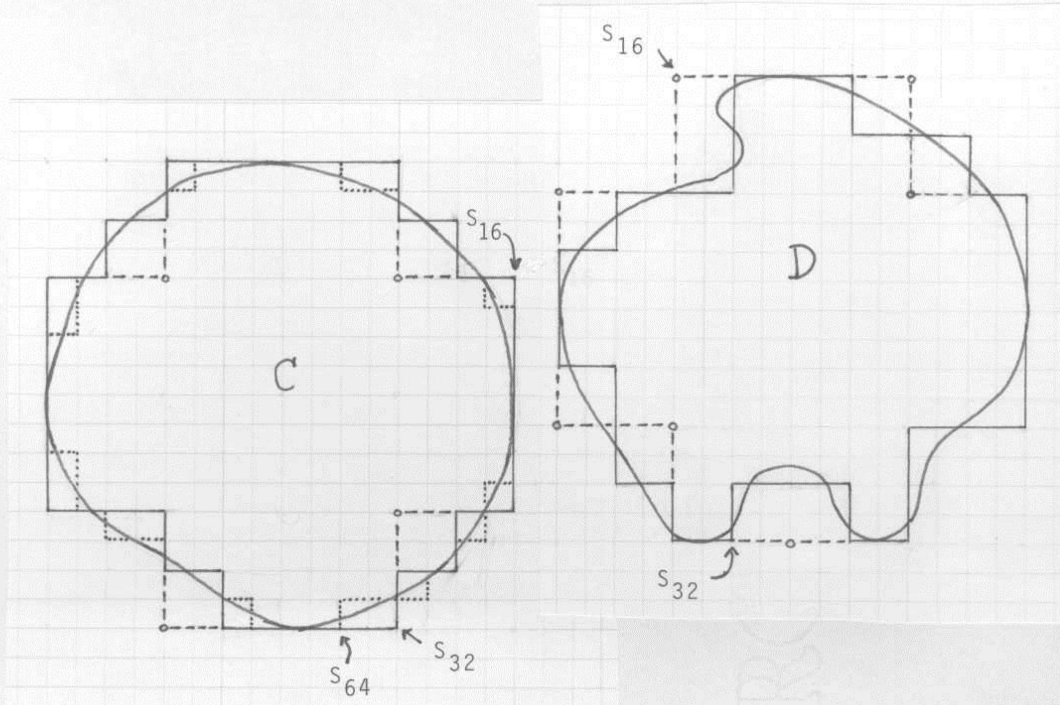
$S_{64}(b) = 1221(321)^2 (31)^2 2312312^3 1(31)^3 2312^5 132(13)^3 12^4 (13)^4 2132$

FIGURE 'REGIONS A AND B, WITH SOME OF THEIR SHAPE NUMBERS

The arrows on the figures signal the beginning of the string of order 32 or 64.

The distances among these Bshapes (ultradistances that measure Bshape) are given in the distance matrix later in the report.

The distance between a and b is 1/16 = 0.0625



$S_4(c) = 1^4 \qquad S_8(c) = (12)^4 \qquad S_{16}(c) = (1213)^4$

$S_{32}(c) = 122(13)^2 212^3 (13)^2 12^3 (13)^2 12^3 (13)^2$

$S_{64}(c) = 122(13)^2 (213)^2 12312^3 (132)^2 12(31)^2 2^4 (132)^2 12(31)^2 2^5 132$
$(13)^2 1223$

$S_4(d) = 1^4 \qquad S_8(d) = (12)^4 \qquad S_{16}(d) = (1213)^4$

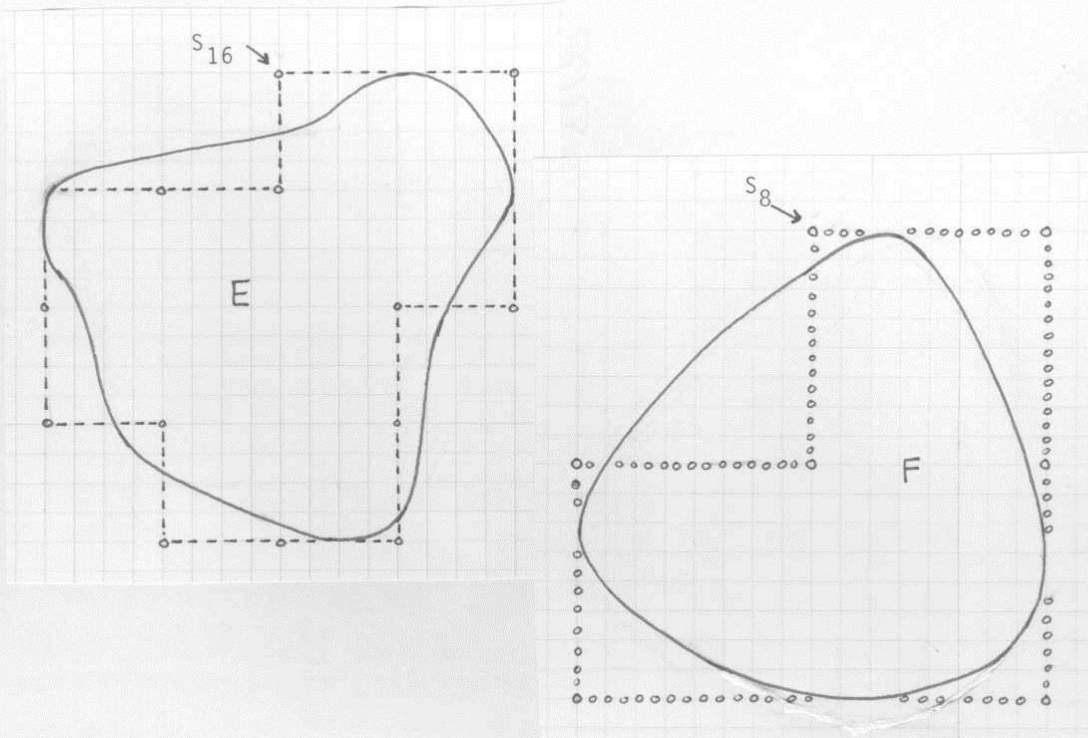
$S_{32}(d) = 1131231213123212132131 2^3 123211323(13)^3 2132$

$S_{64}(d) = 12123 2^3 32121(31)^2 231231 2^3 131231223 2^3 12221321(31)^4$
$2^4 (13)^3 2132$

FIGURE 'REGIONS C AND D, WITH SOME OF THEIR BSHAPE NUMBERS'

The similarity matrix for the bshapes of these regions is found in the following pages.

C and D have a degree of similarity equal to 16; $c \approx_{16} d$.



$S_4(e) = 1^4$ $S_8(e) = (12)^4$ $S_{16}(e) = 1212132121312123$

$S_{32}(e) = 121231223(1213)^2 213221221313123$

$S_4(f) = 1$ $S_8(f) = 11(21)^2 13$

$S_{16}(f) = 113(122)^2 131(13)^2$

FIGURE 'REGIONS E AND F, WITH SOME OF THEIR BSHAPE NUMBERS'

The similarity tree that arranges these regions according to their likeness is given in the next page.

The degree of similarity between e and f is 4.

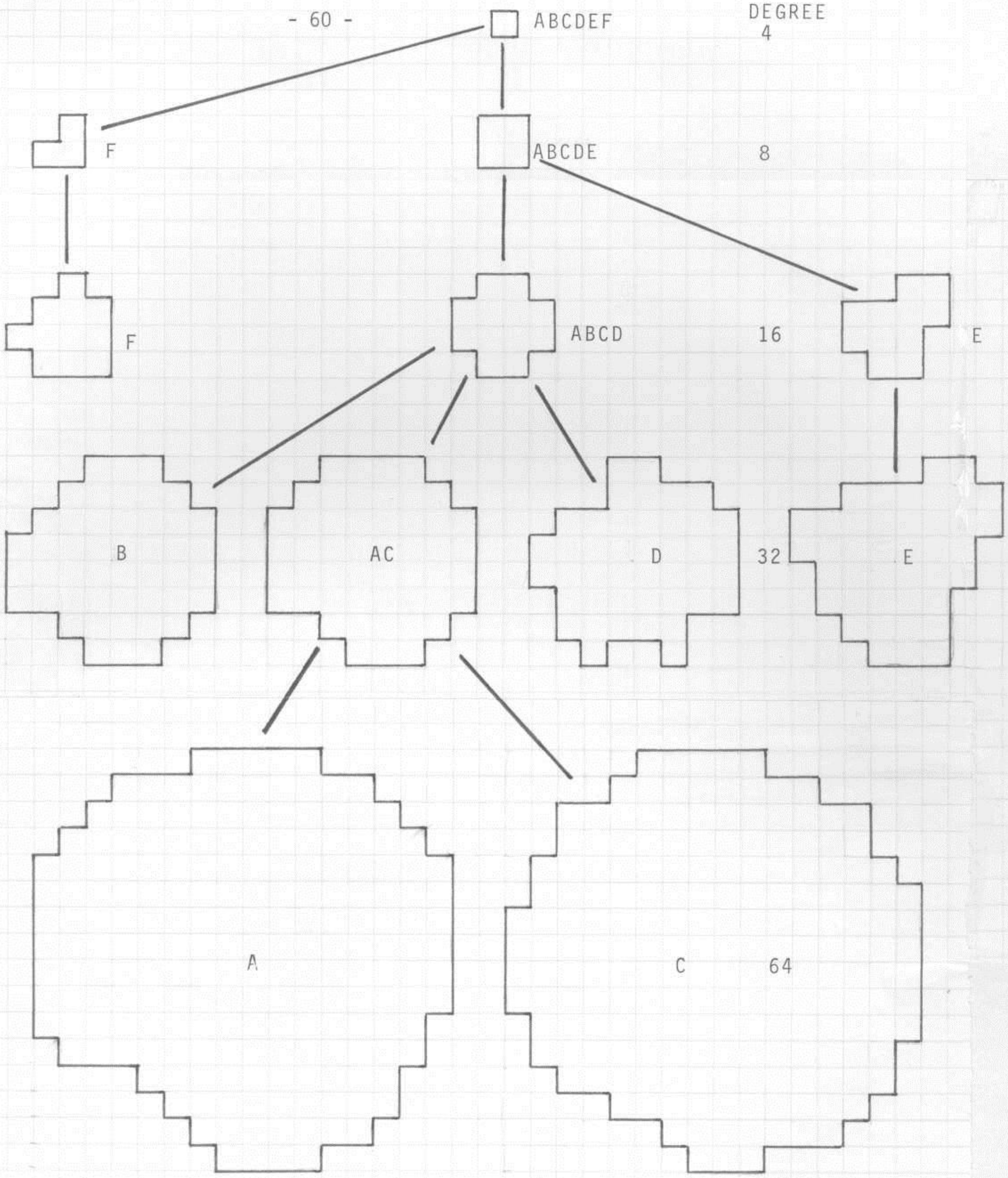


FIGURE 'SIMILARITY TREE FOR THE BSHAPES OF REGIONS A TO F'
These regions were shown in previous pages. The tree shows that the degree of similarity between B and E is 8, but between B and C is 16.

	A	B	C	D	E	F
A	∞					
B	16	∞				
C	32	16	∞			
D	16	16	16	∞		
E	8	8	8	8	∞	
F	4	4	4	4	4	∞

TABLE 'SIMILARITY MATRIX FOR THE BSHAPES OF REGIONS A-F'

Notice that $a \approx_{16} d$, $c \approx_{16} d$ but $a \approx_{32} c$.

	A	B	C	D	E	F
A	0	1/16	1/32	1/16	1/8	1/4
B		0	1/16	1/16	1/8	1/4
C			0	1/16	1/8	1/4
D				0	1/8	1/4
E					0	1/4
F						0

TABLE 'DISTANCE MATRIX FOR BSHAPES OF REGIONS A TO F'

A and C are very close together (1/32) in Bshape. The region F is quite dissimilar (1/4) in shape to all others.

K = 4	K = 8	K = 16	K = 32
CLASS 1: ABCDEF	CLASS 1: F	CLASS 1: F	CLASS 1: F
	CLASS 2: ABCDE	CLASS 2: ABCD	CLASS 2: B
		CLASS 3: E	CLASS 3: A C
			CLASS 4: D
			CLASS 5: E

TABLE 'EQUIVALENCE CLASSES FOR BSHAPES OF A TO F'

The relation "x is similar to y at degree at least k (k fixed)" (i.e., they are similar at k+i, $i \geqslant 0$) forms the above equivalence classes. Notice that each k partitions the set of shapes in different manner.

The relation "x is similar to y at degree k (k fixed)" is not an equivalence relation.

6. Write a procedure to find the excentricity and basic rectangle from the shape number.

7. A better method is needed to encase the perimeter (region) into a box. Noise could introduce errors in length and position [Guzmán 71; Bribiesca and Guzmán; Freeman and Shapira].

8.A. Refer to problems 1 and 2 of first theory. Of course, given an order (30, say) it is possible to find the best shape number of that order that fits the region, by comparing, in the least squares sense, the region with all the shapes of such number. In this way the existence of a shape number for any order and any region could be guaranteed. I suggest to look for a procedure that avoids many comparisons but still gives back the shape number of order 30. This new method could be slower, since it will be used only when the normal procedure fails. But beware of the fact that this method could produce different shape numbers than those produced by the procedure used in the paper. Both methods are not equivalent.

8.B. In order to make the loops vanish, do not use all orders. For instance, use only orders 4, 6, 8, 10, (this will make all loops of length 2 disappear), ..., or even non-linearly spaced: 4, 6, 10, 16, 24, ...

9. Apply these theories to clustering. Do you want to group 200 figures into 24 classes, according to their shape? Construct their similarity tree, and cut it at a level such that the number of branches cut at that level is approx. 24. You could answer relative likeness questions such as: "Is the difference between a and d larger than the difference in shape between e and f?" The answer could be: "yes, because $a \approxeq_{10} d$ and $e \approxeq_{14} f$." e and f went together longer; They needed a stronger lens (of order 16) to separate them.

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